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Risk management for sovereign financing within a debt sustainability framework

This paper presents a model for analysing debt sustainability by optimizing debt-financing decisions to balance borrowing costs with refinancing risks.

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Abstract

The mix of instruments used to finance a sovereign is a key determinant of debt sustainability through its effect on funding costs and risks. We extend standard debt sustainability analysis to incorporate debt-financing decisions in the presence of macroeconomic, financial, and fiscal risks. We optimize the maturity of debt instruments to trade off borrowing costs with refinancing risk. Risk is quantified with a coherent measure of tail risk of financing needs, *conditional Flow-at-Risk*. A constraint on the pace of reduction of debt stocks is also imposed, and we model the effect of debt stocks on the yield curve through endogenous risk and term premia.

On a simulated economy, we show that the cost-risk and flow-stock trade-offs embedded in issuance decisions are key determinants of the evolution of debt dynamics and are economically significant. Comparing three alternative optimizing strategies and some simple fixed-issuance rules, we also draw lessons on when and why optimizing matters the most. This depends on the risk tolerance level, the size, cost, and maturity of legacy debt, and the sensitivity of interest rates to debt.

Our model quantifies thresholds for the minimum level of refinancing risks and the maximum pace of debt reduction that a sovereign could reach given its economic fundamentals. Going beyond those thresholds is only feasible through adjustments of gross financing needs, and an extension of the baseline model identifies the hot spots for these adjustments, computing their minimum size and optimal timing. Our findings inform policy decisions concerning both official sector borrowing and public finance, with a focus not only on minimizing interest payments but also on managing refinancing risks and increasing debt dynamics.

Keywords: sovereign debt, sustainability, debt financing, optimization, stochastic programming, scenario analysis, conditional Value-at-Risk, risk measures

JEL codes: C61, C63, D61,E3, E47, E62, F34, G38, H63

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1 Introduction

In the aftermath of the 2008 global financial crisis, sovereign debt increased sharply in most advanced economies. Figure 1 makes the point quite clearly. Average public debt, as a percentage of gross domestic product, increased by one third from trough to peak, with almost all countries experiencing a significant increase. In the euro area, public debt rose to about 84% in 2010, which decisively contributed to a sovereign debt crisis in the region, with five countries (Greece, Ireland, Portugal, Spain, Cyprus) requiring external financial assistance. These episodes, and the fact that public debt levels have barely declined (if not increased) since then, have prompted a renewed interest in debt sustainability analysis (DSA) and have led to intense policy discussions concerning, for instance, the most appropriate variable (e.g., debt stock level or annual gross financing needs) and the appropriate thresholds to assess debt sustainability. Amid this debate, it has become clear that standard DSA models, widely used by international institutions like the International Monetary Fund and the European Stability Mechanism need to be strengthened to serve as an early warning tool and to inform about the efforts required to improve the sustainability of public accounts in troubled economies.

In this paper, we extend standard DSA models to incorporate optimal debt-financing decisions for an economy facing uncertain economic growth, interest rates, and fiscal balance. This framework is well equipped to address questions like: How do issuance strategies trade off interest costs and refinancing risks? Through which channels do they influence debt flow and stock dynamics? How do these interactions depend on the stock of legacy debt, risk tolerance, or the sensitivity of interest rates to debt? How and when should gross financing needs be adjusted to render sustainable debt dynamics?

The model determines the maturity of debt-financing instruments to minimize expected debt interest costs subject to constraints on both gross financing needs and debt dynamics. Our constraint on gross financing needs limits the potential refinancing risks that could arise from spikes in funding requirements, while the constraint on debt dynamics sets the minimum (maximum) pace at which the debt-to-GDP ratio may decrease (increase) over time. Importantly, following the risk management literature, we specify both constraints in terms of the conditional Value-at-Risk (CVaR) measure, which focuses on the expected values of the distribution tails. This approach, engrained in the day-to-day risk management of financial institutions, is particularly appropriate for debt sustainability analysis given the multiple sources of uncertainty at play. In the model, we represent this uncertainty by a discrete time- and state-space scenario tree, and use stochastic programming on this tree to optimize the debt financing strategies. Moreover, we address the fact that a sovereign's debt sustainability outlook influences its market risk premium and the slope of its yield curve. To capture this link, the *risk* and *term premia* in our model change endogenously with the sovereign's debt level, which in turn affect issuance decisions and, consequently, the debt level, creating a feedback loop.

The main innovations of the model is the optimization of debt financing decisions with endogenous interest rates, the simultaneous treatment of debt stock and flow, the introduction of a risk measure in the analysis, and the identification of *hot spots* where gross financing needs may have to be adjusted to meet policy targets. Our model looks at multiple aspects of sustainability and does so for short, medium, and long-term horizons.

Testing the model on a simulated economy, we learn that there are two marked trade-offs in choosing the optimal issuance strategy: those between costs and risks, and between flow and stock dynamics. In particular, reducing refinancing risks tends to foster long-

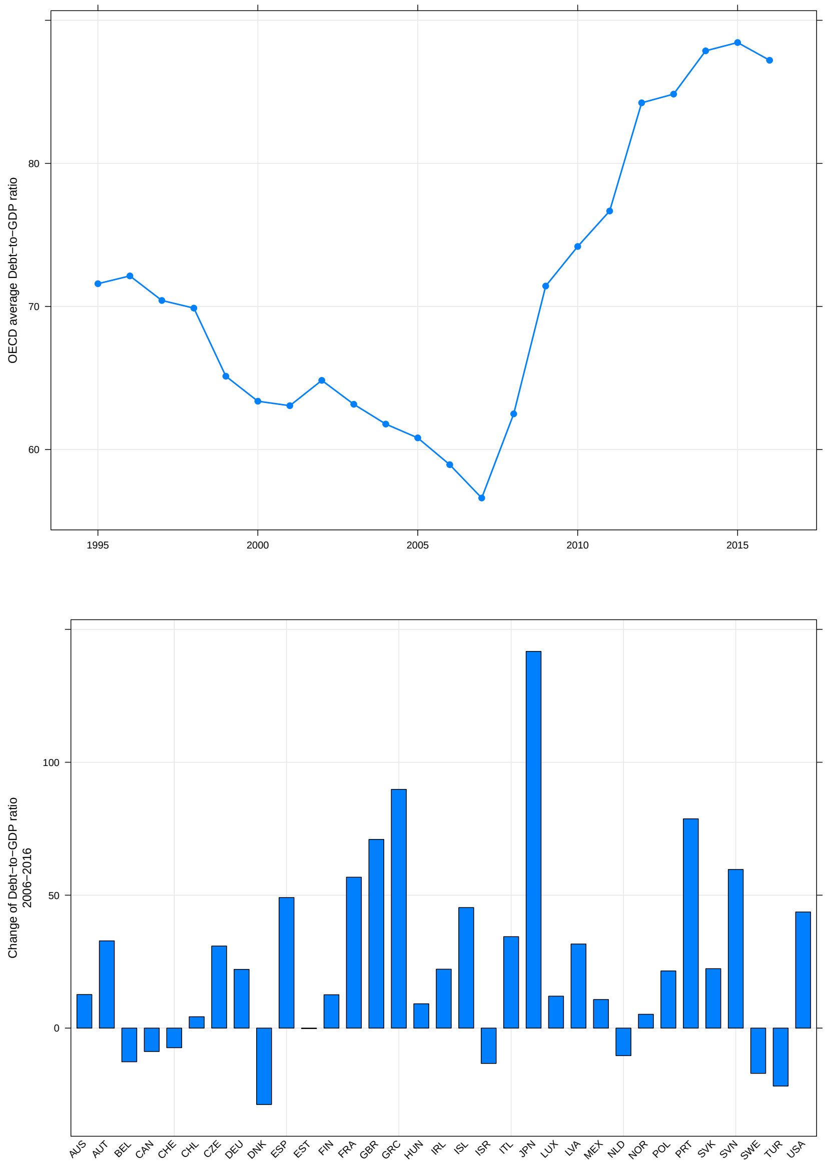


Figure 1 – Debt growth of OECD countries. (Data: OECD, General government debt indicator doi 10.1787/a0528cc2-en, accessed Dec. 16, 2017.)

term issuances and lead to smoother debt flow dynamics. Long-term financing, however, is more expensive than short-term, and weighs on debt stock dynamics via higher effective interest rates. An important contribution of our paper, especially relevant for policy work, is to quantify these trade-offs. We find that they are economically significant and have a material (non-linear) impact on the relevant variables for debt sustainability analysis. For instance, moving from the risk minimization to cost minimization, in a calibrated economy, implies a 40% reduction in interest payments with an increase of refinancing risks by 10% of GDP. The difference on debt dynamics is, on average, 9% of GDP reduction in stock and 8% increase in gross financing needs.

We compare debt flow and stock dynamics for optimal strategies with different degrees of flexibility and for some simple rules with fixed debt issuance in one (or more) maturities. Naturally, optimal strategies outperform simple rules, and the more flexibility the better.

More importantly, we find that the benefits of optimizing are economically meaningful (both in terms of costs and risks), and that they are relatively larger when risk tolerance is lower, the stock of legacy debt is larger or its maturity shorter, and funding costs are more sensitive to debt dynamics. Model performance is consistent with results from the economics literature on taxation and deficit smoothing, and "gambling for redemption" (Conesa and Kehoe, 2015) of highly indebted countries.

Another important contribution of our model is that it may quantify the minimum level of refinancing risks and the maximum pace of debt reduction that a sovereign could reach (following optimal issuances strategies) given its economic fundamentals. These thresholds could provide a benchmark for assessing debt sustainability, and knowing them would advise feasible policy targets. In an extension of the model, we evaluate how a sovereign can go beyond those thresholds through adjustments of gross financing needs, using either domestic (e.g., fiscal effort, privatization) or external (e.g., official sector support, debt restructuring) resources. In particular, we identify the *hot spots* where gross financing needs may be excessive, and compute the minimum size and optimal timing of the adjustments so that policy goals are met. Our results suggest that the sooner these additional adjustments are implemented the better, and we are able to measure the economic cost of delays. This feature of the model is consistent with the findings from the stylized model of Blanchard et al. (1990) that "delaying adjustment substantially affects the size of the needed policy action", and it may be helpful in designing future official-sector financial assistance programmes

The remainder of the paper is organised as follows. Section 2 reviews the three strands of the literature related to our paper. The basic layout of the model is described in Section 3, and the model is developed in Section 4. Section 5 discusses the calibration used in our simulations, and Section 6 presents the model's main qualitative and quantitative findings on the interaction between debt-financing decisions and debt sustainability. In Section 7, we extend our baseline model to identify hot spots and calculate the minimum adjustments that improve a sovereign's debt sustainability outlook beyond that implied by its assumed fundamentals. Finally, Section 8 identifies areas for further work. One appendix presents some alternative risk measures that we also explored and that deserve further study.

2 Relation to existing literature

Our paper draws from three strands of literature. First, the economics literature provides several stylized models and empirical investigations of salient features of sovereign debt financing and debt sustainability. Second, the asset and liability management literature addresses the perennial problem of trading off risks and rewards, and provides the risk measure. Third, management science and operations research provide stochastic programming models for planning under uncertainty. Our overarching contribution is to integrate many of the innovations that were developed, in isolation, in these strands of the literature into a comprehensive optimization framework that addresses debt sustainability from a risk management perspective, and it is sufficiently granular and flexible to provide meaningful qualitative and quantitative insights for policy makers.

Economics of sovereign debt

The literature on what determines sustainable debt levels is extensive. D'Erasmo et al. (2016) summarize the various generations of models, including Blanchard et al. (1990) debt sustainability indicators and Bohn (1995) on fiscal rules. Our paper adds to recent contributions which model an active fiscal policy maker. Rather than allowing for endogenous default[1], we focus on the debt management side of policy decisions.[2]

A number of recent contributions to this literature focus, as we do, on the role of the maturity structure and place the trade-off between borrowing costs and refinancing risks at a center stage.[3] Cole and Kehoe (2000); Conesa and Kehoe (2014) find that sovereigns faced with a self-fulling refinancing crisis should lengthen debt maturity, whereas countries that are in a recession gamble for redemption by shortening the debt maturity (Conesa and Kehoe, 2015). Barro (2003) argues that a tax-smoothing objective leads to contingent and long-term debt financing strategies, and Angeletos (2002) argues that governments can issue long-term debt to invest in short-term reserves in order to insulate public debt from interest rate risk and smooth refinancing needs. In Niepelt (2008), the presence of sovereign risk leads towards shorter maturities when debt issuance is high and or output is low. Similarly, Arellano and Ramanarayanan (2012); Broner et al. (2013) show that when interest rates rise, maturity shortens. Relatedly, Aguiar et al. (2016) show that, in a debt crisis, it is optimal to switch to short-term financing which provides better incentives to repay.

It is noteworthy that the recommendations for an appropriate debt maturity differ among these papers. This does not reveal any fundamental disagreements in the literature, but is, instead, a natural outcome of models seeking strategic recommendations under different stylized assumptions. Also, some of these assumptions may not be suitable for practical applications. For instance, Arellano and Ramanarayanan (2012); Hatchondo and Martinez (2009) assume debt financing using consol bonds (perpetuities) that pay an infinite stream of coupons which decreases at a constant rate, and Chatterjee and Eyigungor (2012) analyze long-term debt contracts that mature probabilistically. We add to this literature a normative model using a richer and more realistic set of instruments, that incorporates in a common framework multiple considerations addressed independently in previous studies. The model generates *dynamic* financing strategies, tailored to the problem at hand, and with sufficient granularity that permits a broad range of maturities not restricted to the dichotomous choice between short-term treasuries and consols.

Recent empirical contributions focus on the role of debt-related cashflows for debt sustainability and market access. Dias et al. (2014) show that identical debt stocks can have very different cashflows, Weder di Mauro and Schumaker (2015) show that different amortization schedules and varying interest rates make a given stock of debt mean very different things for the Greek debt flow, Irwin (2015) looks for the right definition of public debt to be used for assessing sustainability, Gabriele et al. (2017) show that to

[1]See Aguiar et al. (2016); Arellano and Ramanarayanan (2012); Bai et al. (2015); Chatterjee and Eyigungor (2012); Conesa and Kehoe (2014); Hatchondo et al. (2016); Hatchondo and Martinez (2009); Mihalache (2017).

[2]By excluding endogenous default from our analysis, we are able to model a rich environment for debt financing decisions, including a broad range of maturities, which significantly enriches the trade-off between financing costs and refinancing risks.

[3]Other aspects of recent sovereign debt literature include sustainability in Monetary Unions (Aguiar et al., 2014), the role of inflation (Aguiar et al., 2013, 2016), and the role of structural reforms accompanying official sector borrowing (Muller et al., 2015).

understand borrowing costs we must jointly consider debt stock and refinancing needs. Bassanetti et al. (2016) show that changes of debt stock are an important driver of a country's borrowing costs. Our paper supplements this literature by modeling the tension between stock and flow within the framework of DSA, using a debt accumulation equation, in which excessive refinancing needs and non-decreasing debt level are penalized (IMF, 2013; Weder di Mauro and Schumaker, 2015). We also show how debt-financing strategies influence stock-flow dynamics, and how all of them are affected by the feedback loop between debt stocks and interest rates.[4] On the latter, Broner et al. (2013); Engen and Hubbard (2004); Paesani et al. (2006) shed light on how the yield curve of government securities responds to a country's fiscal position. Qiang and Phillippon (2005) show that debt financing rates are determined by a risk-free rate, a risk premium for the idiosyncratic risk of the sovereign, and term premia for debt of different maturities. These premia depend on both debt stock (Paesani et al., 2006) and refinancing requirements (Bassanetti et al., 2016; Gabriele et al., 2017). We use this literature to calibrate a functional response of interest rates to debt dynamics, and integrate it into our model.

Risk and asset/liability management

Asset and liability management of financial institutions hinges on trade-offs between risks and expected rewards under uncertainty (Mulvey and Ziemba, 1998; Zenios and Ziemba, 2007). Regulated industries, such as insurance and banking, developed standards on how to measure and limit risks in Basel III and Solvency II, respectively. There is a trade-off between risks and rewards, and, dating back to the pioneering work of Markowitz (1952) on portfolio selection, these trade-offs are quantified through efficient frontiers. Trade-offs in the context of sovereign debt management have been recognised by Missale (1997, 2000), where he relates the policy implications of tax or deficit smoothing to a trade-off between minimization of the expected cost of debt servicing and of budgetary risk, and concludes that the optimal smoothing approach follows from a specification giving all the weight to risk minimization. Bolder (2003) developed a simulation framework to study the trade-offs associated with different financing strategies, and Velandia (2018) describes an asset and liability framework for sovereigns. In the context of debt sustainability analysis, the quantification of uncertainty has only recently received systematic attention by international organizations and academia (Celasun et al., 2006; Consiglio and Zenios, 2016; Guzman and Lombardi, 2018). Our contribution to this literature is in the use of discrete scenario trees with risk and term premia to reflect the endogeneity of debt (re)financing rates, and the introduction of a risk measure with the associated parametric optimization model to trace the trade-off.

Risk measures in univariate DSA were used by Barnhill and Kopits (2003), who model the probability of a negative net worth position for the government and obtain Value-at-Risk (VaR) for the balance sheet of the consolidated public sector. In our work we adapt the conditional Value-at-Risk (CVaR) which is used in Basel III. CVaR is a tail measure and this is important in the context of DSA, since un-sustainability are rare extreme events giving rise to fat tails, and we argued elsewhere (Consiglio and Zenios, 2015) that the "devil is in the tails". This risk measure is theoretically grounded in the properties of coherence (Artzner et al., 1999). For a discussion of VaR and CVaR see, e.g., (Jorion, 2006; Zenios, 2007, pp. 58–63). In a seminal contribution Rockafellar and Uryasev (2000,

[4](Bohn, 1990, p. 1218) recognizes that "if debt management affected interest rates, the qualitative nature of the government's optimization problem would change significantly".

2002) showed that CVaR can be minimized using linear programming, and we use this key property for a tractable formulation of the model.

Planning under uncertainty

We adopt the discrete time-space, discrete state-space modeling framework of multi-period stochastic programming. Stochastic programming dates back to Dantzig (1963) and received renewed attention in the 1980s with the development of solution algorithms (Birge and Louveaux, 2011; Kall and Wallace, 1993), and advances in parallel computer architectures (Censor and Zenios, 1997; Hiller and Eckstein, 1993) that facilitated the solution of large-scale problems. Applications in finance have proliferated since the 1990s.[5]

Stochastic programming models were developed for the Turkish Ministry of Finance (Balibek and Köksalan, 2010) and for the Italian Treasury (Consiglio and Staino, 2012). These earlier works deal with the short-term problem of public debt management to optimize the cost of debt issuance. Consiglio and Zenios (2016) used the CVaR risk measure in a sovereign debt stock model to carry out sensitivity analysis for debt sustainability. These works do not consider debt flow dynamics, or economic and fiscal shocks, but only an exogenous stochastic yield curve. These are the precursors to our model.

3 Layout of the model

3.1 The economic problem

We consider a sovereign that at period t is endowed with output Y_t, runs a primary balance PB_t, and owes a stock of debt D_{t-1}. The sovereign's *gross financing needs* are given by the *flow dynamics*

$$GFN_t = i_{t-1}D_{t-1} + A_t - PB_t, \tag{1}$$

where i_{t-1} is the *effective nominal interest rate* on debt at $t-1$, and A_t is the amortization schedule corresponding to the amount of D_{t-1} that matures at t.[6]

To finance its needs the sovereign choses from J debt financing instruments of different maturities.[7] At t the sovereign makes *debt financing decisions* to issue $X_t(j)$ nominal amount of instrument j. The *debt financing equation* satisfies

$$\sum_{j=1}^{J} X_t(j) = GFN_t. \tag{2}$$

[5]They include models for dynamic asset allocation (Hibiki, 2006; Mulvey and Vladimirou, 1992), personal financial planning (Berger and Mulvey, 1998; Consiglio et al., 2007; Dempster and Medova, 2011), defined-benefits and defined-contributions pension plans (Hilli et al., 2007; Mulvey et al., 2008; Ziemba, 2016), insurance with or without minimum guarantees (Carinõ and Ziemba, 1998; Consiglio et al., 2006; Mulvey et al., 2000), mortgage portfolio management (Zenios et al., 1998), corporate bonds (Jobst et al., 2006), and for asset and liability management of large institutions (Mulvey and Ziemba, 1998; Zenios and Ziemba, 2007). Stochastic programming also found applications in pricing derivative securities in incomplete markets (King, 2002; King et al., 2005), including the pricing of sovereign contingent debt (Consiglio and Zenios, 2018).

[6]Eqn. (1) does not include one off items and stock-flow adjustments, but these can be easily added.

[7]The model allows instruments that are not differentiated only by maturity. They can be denominated in foreign currencies or be contingent debt such as GDP-linked bonds or sovereign CoCos.

The interest on issued debt is determined by the market risk-free rate plus premia idiosyncratic to the sovereign and the chosen maturities. We assume endogenous premia, as a nonlinear function of debt-to-GDP ratio, $d_t = \frac{D_t}{Y_t}$ and of the maturity of the issued instrument (Engen and Hubbard, 2004; Paesani et al., 2006). The interest rate for instrument j issued at t is given by

$$r_t(j) = r_{ft} + \rho(d_t, j). \tag{3}$$

$\rho(d, j)$ is a function that captures both the *risk premium* as a monotonically increasing function of debt ratio, and *term premia* for different maturities; see section 5.3 for calibration.

The effective interest rate depends on the issued debt. The vector of issuance of debt of all types j at each time t, denoted by X, determines the interest rate, and $i \doteq i(X)$. The *effective interest rate* paid by the sovereign is given by[8]

$$i_t = \frac{i_{t-1}(D_{t-1} - A_t) + \sum_{j=1}^{J} r_t(j) X_t(j)}{D_t}. \tag{4}$$

In this paper we model the optimal choice of the debt financing variables X. These variables consequently determine the dynamics of debt, but the debt level determines risk and term premia which, in turn, influence the maturities to be issued. This endogeneity creates a *feedback loop* $X \to D \to r \to X$ that links stock and flow not only through quantities but also through prices. This feedback from debt stocks into interest rates, which may give raise to virtuous and/or vicious circles, is an important feature of our model which critically influence an issuer's debt-financing decisions.

3.2 Modelling uncertainty

We model uncertainty using a discrete multi-period *scenario tree*, see Figure 2. Time steps are indexed by $t = 0, 1, 2, \dots T$, where 0 is here-and-now, and T is the risk horizon. Data are indexed on the tree by a set of *states* $\mathcal{N}_t$, with each $n \in \mathcal{N}_t$ representing a possible state of the economy at time t. $\mathcal{N}$ denotes all possible states during the risk horizon. The number of states at t is N_t, and the total number of states is N. Not all states at t can be reached from every state at $t-1$, and $\mathcal{P}(n)$ denotes the set of states on the unique *path* from the *root state* 0 up to n. Each path that leads to a terminal state $n \in \mathcal{N}_T$ is a *scenario*. The unique ancestor of n is denoted by $a(n)$, with $a(0) = 0$. We also need a function $\tau(n)$ to identify the time period of state n on a path, i.e., $\tau(n) = t, \tau(a(n)) = t-1, \tau(a(a(n))) = t-2$ and so on. For any state n at t, all information at states m on the path $\mathcal{P}(n)$ is known since $\tau(m) < t$.

Exogenous and endogenous variables of the problem are indexed by states n that belong to set $\mathcal{N}_t$. The values of all exogenous variables are known for each state, whereas endogenous variables are determined by the optimization model and take state-dependent values. The conditional probabilities of states $n \in \mathcal{N}_t$ are denoted by π_t^n, and p^n denote unconditional probabilities of states $n \in \mathcal{N}$ except the root. Scenario probabilities are the unconditional probabilities of states $n \in \mathcal{N}_T$.

[8]For simplicity, this recursive expression assumes that the effective interest rate for D_{t-1} and A_t is the same. In practice, this does not need to be the case, and this is fully accounted in all simulation exercises in the paper.

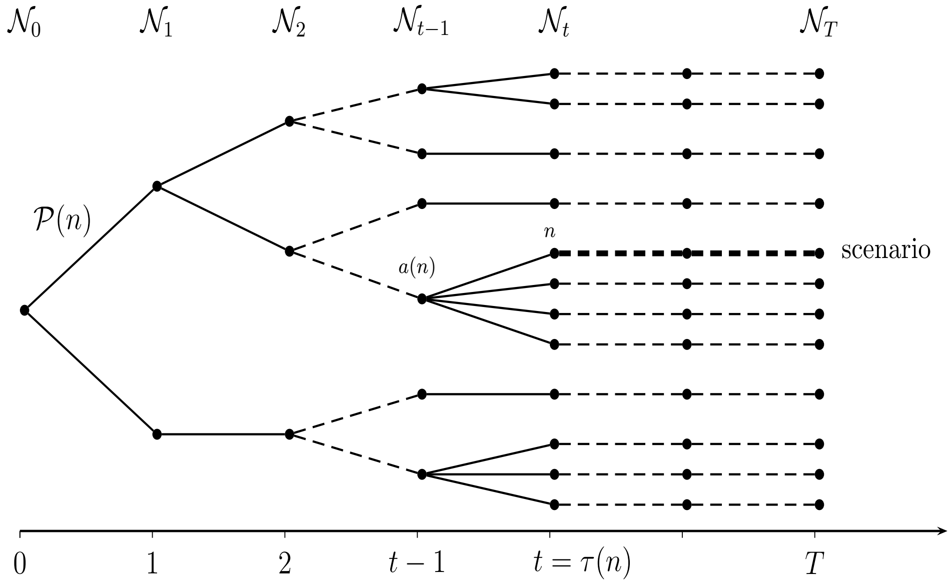


Figure 2 – A scenario tree.

The discrete multi-period and multi-state tree setup allows us to model the term structure of debt with issues of different maturities, and the clustering of maturities around specific dates with the associated refinancing risk due to excessive spikes of financing needs, using a level of granularity that is not restricted to the dichotomous choice between short-term treasuries and consols. On the tree we specify debt financing decisions that can be both time- and state-dependent and represent the stochastic dynamics of both debt stock and flow. This allows us to define risk measures of the (temporal) distributions of stock and flow dynamics. We then formulate a model to optimize debt financing strategies to trade off cost with risk and look at the risk of violating some sustainability conditions, such as debt flow exceeding some empirically observed market thresholds or increasing debt stock. At each node of the tree we also compute a calibrated non-linear function of the risk and term premia, thereby endogeneizing interest rates.

4 The optimization model

We pose the economic problem of the sovereign as an optimization model on the scenario tree. Specifically, we discuss the objective function, define the debt financing decision variables, and set the constraints for a baseline model where the sovereign faces uncertain exogenous economic output and primary balance, and uncertain refinancing rates that are endogenous.

The baseline model captures the most important considerations influencing the decisions of a sovereign issuer. In particular, the trade-off between short-term and long-term costs, and the assessment of future refinancing risks arising from shocks in the economy and in capital markets. The model is sufficiently flexible to accommodate market-specific considerations, such as smooth or boundary conditions on the issuance strategy across maturities arising from past issuances.

We introduce constraints on debt flows and stocks. Debt flows give a vulnerability signal at any risk horizon, whereas debt stock dynamics reflect long-term solvency, and these variables are linked through endogenous interest rates. We set the constraint on debt flows in terms of a risk measure that quantifies the tail risks in the stochastic distribution of this variable. As for debt stocks, we require that they follow a non-increasing path to limit insolvency risks. Naturally, relaxing the constraint on one variable improves the optimal value for the other variable.

4.1 Objective function

We consider a sovereign issuer that tries to minimize the expected interest payments on its debt, subject to a constraint on the level of refinancing risks incurred. We measure gross financing needs, like debt stock, as a proportion of a country's economic output to account for output uncertainty, and denote this random variable by *gfn*. Using $\Psi(\cdot)$ to denote the risk measure, the optimization problem is expressed as:

$$\text{Minimize}_X \qquad \sum_{n\in\mathcal{N}} p^n NIP_t^n \tag{5}$$

$$\text{s.t.}$$

$$\Psi(gfn) \quad \leq \quad \omega. \tag{6}$$

The objective function minimizes the expected *net interest payments* (NIP) faced by the sovereign, which is the single most relevant variable for the treasury. (Recall that summation over $n \in \mathcal{N}$ is equivalent to summation over $n \in \mathcal{N}_t$, for all $t = 0, 1, 2, \ldots T$, so the summation in the objective function is well defined.) Interest payments consist of interest service payments on legacy debt I_t^n, plus service payments on debt created endogenously by the financing decisions on the path leading to n. To trace service payments on endogenously created debt requires some ingenuity to exploit the tree structure. Let $CF_t^n(j,m)$ denote the nominal amount of interest payments due at state n of period t per unit $X^m_{\tau(m)}(j)$ issued at state m of period $\tau(m)$ on the path $\mathcal{P}(n)$. This variable can be computed exogenously from scenarios of the term structure of interest rates and the terms of the instrument; if the yield curve depends on debt stock (eqn. 3), this variable becomes endogenous. The net interest payments are then given by

$$NIP_t^n = I_t^n + \sum_{m\in\mathcal{P}(n)} \sum_{j=1}^{J} X^m_{\tau(m)}(j)\, CF_t^n(j,m). \tag{7}$$

Net interest payments minus interest on legacy debt is what the sovereign controls through debt financing decisions. NIP/D is the effective interest rate of debt (cf. eqn. 4).

The constraint in the optimization problem above sets limits on the acceptable risk level (ω) on gross financing needs. This can be interpreted as the sovereign's tolerance towards potential refinancing risks. The random variable *gfn* takes values indexed by the state $n \in \mathcal{N}_t$ at time t,

$$gfn_t^n = \frac{GFN_t^n}{Y_t^n}, \tag{8}$$

where Y_t^n is the state-dependent economic output. We denote by *gfn* the random variable over all states $\mathcal{N}$, and by gfn_t over states $\mathcal{N}_t$ at t.

4.2 Decision variables

The debt financing decisions on the tree are denoted by $X_t^n(j)$ and the debt financing equation (2), for all states $n \in \mathcal{N}_t$, and times periods $t = 0, 1, 2, \ldots T$, becomes

$$\sum_{j=1}^{J} X_t^n(j) = GFN_t^n. \tag{9}$$

We can denote debt financing decisions using proportional weights $w_t^n(j)$ for instrument j in state n at time t, and write the debt financing equation as

$$\sum_{j=1}^{J} w_t^n(j) = 1, \tag{10}$$

$$w_t^n(j) = \frac{X_t^n(j)}{GFN_t^n}. \tag{11}$$

We now envisage three possible levels of granularity in the Treasury's optimal decision. In the simplest case, the Treasury would be restricted to set weights $w(j)$ which are time- and state invariant. In this set up, the Treasury would search for the best weights across financing instruments considering that these will be constant for all periods and all states. We call this a *fixed-mixed strategy*, and it results in simple *rules* for debt financing.

In the second case the Treasury specifies state-invariant but time-dependent weights $w_t(j)$. In other words, debt is financed using weight allocations in the available instruments that adapt with time but are fixed for all the states at each period. We call this an *adaptive fixed-mix strategy*.

Finally, in the most flexible case, the Treasury specifies both time- and state-dependent optimal weights $w_t^n(j)$. This allows the issuer to implement a decision, wait to observe the state at the next time period, implement the optimal decision for that state, and wait again. We call this a *dynamic strategy*. It has more degrees of freedom than the previous two strategies, and this allows the Treasury to achieve better results. Our investigations later on show that optimal solutions improve significantly when going from simple rules, to adaptive fixed-mix, and to dynamic strategies. Nonetheless, the richer dynamic strategy also poses some practical problems that favour the use of adaptive fixed-mix strategies. Most importantly, real-life evidence shows that, while Treasuries enjoy some flexibility in setting their issuance strategies, in practice they face many demand and supply constraints that prevent them from implementing fully dynamic strategies. Indeed, most Treasuries tend to pre-commit to annual issuance plans some months before the start of the year. Not only that, but they also try to avoid any material change in those plans over the course of the year, mostly for reputational reasons.

Therefore, we consider adaptive fixed-mix as the appropriate strategy for sovereign debt financing, and this is our baseline model. Financing with simple rules is typical of standard DSA and we use it as benchmark in some experiments. Dynamic strategies are the theoretical best and we use them to get a lower bound on cost and risks.

Remark 1. In the dynamic model, decisions are made at $t = 0$ based on all available information at the root state of the scenario tree, including conditional expectations about future uncertain information. As new information arrives at subsequent time periods, the model makes *recourse* decisions. The decision $X_t^n(j)$ is *adapted* to state n from the information at the ancestor state $a(n)$. The tree structure precludes decisions from being adapted to states that have not yet been observed, satisfying the *non-anticipativity* property of stochastic programming. Adaptation and non-anticipativity imply that stochastic programming prescribes implementable policies, without clairvoyance.

Remark 2. In linear scenario structures, i.e., scenario fans instead of scenario trees (Celasun et al., 2006), all information is revealed at $t = 1$. A dynamic strategy would adapt to information that is assumed to be known after $t = 1$, but in practice it will not be so. Multi-period stochastic programs on linear scenario structures are endowed with clairvoyance, while a model with adaptive fixed-mix and simple rules is not. Adaptive fixed-mix models and simple rules are the only ones possible on linear scenario structures.

Remark 3. The optimization model for adaptive fixed-mix and for simple rules is non-linearly constrained. Special purpose algorithms to solve large-scale applications are available (Maranas et al., 1997). We take advantage of the special structure of the model to obtain a starting solutions using linear programming and the non-linearly constrained problem is then solved with relative efficiency.

4.3 Constraints

Risk measure and the debt flow constraint

We use the coherent risk measure conditional Value-at-Risk.[9] In particular, we consider the expected value of the tail of the distribution of *gfn* for confidence level α. This is the aggregate conditional Value-at-Risk of debt flow (CFaR) over the tree, to be distinguished from the risk measure of the debt flow at each time period. Under this approach the risk function in eqn. (6) is defined as

$$\Psi(gfn) \doteq \mathbb{E}\left(gfn \mid gfn \geq gfn^{\diamond}\right), \tag{12}$$

where $gfn^{\diamond}$ is the right α-percentile of the aggregate gross financing needs, i.e., it is the lowest value $gfn^{\diamond}$ such that the probability of the gross financing needs less or equal to $gfn^{\diamond}$ is greater or equal to α. $gfn^{\diamond}$ is the Value-at-Risk of aggregate debt flow, and we use $gfn^{\diamond\diamond}$ to denote aggregate CFaR. These definitions are illustrated in Figure 3.

Following Krokhmal et al. (2002); Rockafellar and Uryasev (2000, 2002), we compute aggregate CFaR on the tree using the following linear system, for all states $n \in \mathcal{N}$,

$$gfn^{\diamond\diamond} = gfn^{\diamond} + \frac{1}{1-\alpha} \sum_{n \in \mathcal{N}} p^n z^n \tag{13}$$

$$z^n \geq gfn_t^n - gfn^{\diamond} \tag{14}$$

$$z^n \geq 0, \tag{15}$$

and the risk constraint becomes

$$gfn^{\diamond\diamond} \leq \omega. \tag{16}$$

ω is the sovereign's tolerance towards potential refinancing risks captured by the tail of gross financing needs. Since $n \in \mathcal{N}$ is equivalent to $n \in \mathcal{N}_t$, for all $t = 0, 1, 2, \ldots T$, it follows that eqn. (14) with time indexed gfn_t^n but time independent z^n, is well defined.

[9] In Appendix A we discuss alternative risk measures: worst-case for stress testing, and risk-neutral.

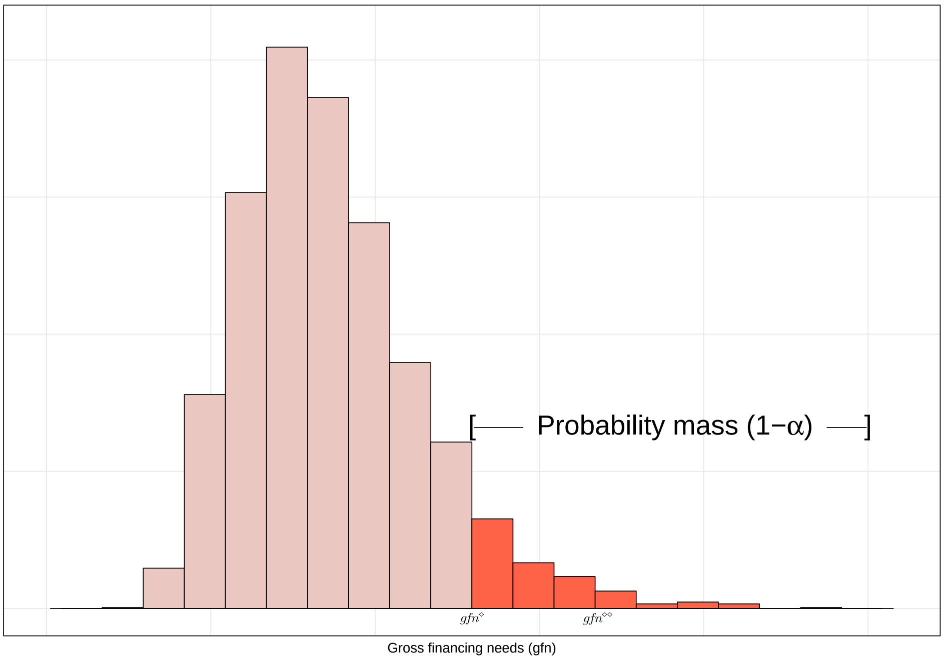


Figure 3 – Distribution of aggregate gross financing needs (gfn), illustrating Flow-at-Risk ($gfn^{\diamond}$) and conditional Flow-at-Risk ($gfn^{\diamond\diamond}$) at confidence level α.

Remark 4. Bounding the aggregate CFaR by a threshold does not guarantee that CFaR will be below the threshold at each time period. It may exceed the threshold at some time t' at the α confidence level of the distribution $gfn_{t'}$. t' will be a *hot spot* for debt flow. In the empirical work we consistently found that the aggregated formulation also limits the dis-aggregated risk. However, if some country faces a spike of legacy debt, that could originate a hot spot and the dis-aggregated measure may exceed the threshold. If the threshold is exceeded at some hot spot(s) we can impose CVaR constraints at the hot spot to shape risk. This requires computing the conditional Flow-at-Risk (CFaR) for gross financing needs at each t, defined by

$$\Psi(gfn_t) \doteq \mathbb{E}\left(gfn_t \mid gfn_t \geq gfn_t^{\diamond}\right), \tag{17}$$

where gfn_t is the random variable of gross financing needs at t, $gfn_t^{\diamond}$ is the right α-percentile, or Flow-at-Risk, and CFaR is denoted by $gfn_t^{\diamond\diamond}$. The disaggregated risk measure can also be formulated based on the work of Rockafellar and Uryasev (2002) to shape the risk profile at each time period, or, at select hot spots, as used by Jobst et al. (2006) for credit portfolios.

Debt stock constraint

We add a debt stock constraint to model (5)–(6),

$$\frac{\partial d}{\partial t} \leq \delta. \tag{18}$$

If $\delta = 0$, the debt stock constraint implies that the debt-to-GDP ratio should be non-increasing over time. In contrast, a negative δ imposes a minimum annual debt reduction pace. Finally, debt could increase over time up to an annual bound if δ was positive. For highly indebted countries, debt stock should be decreasing and δ is set to some negative value specifying the desirable pace of decrease, whereas debt increases may be allowed for low debt countries by using positive parameter. We compute changes of d on the tree by

$$\Delta_t^n = d_t^n - d_{t-1}^{a(n)}. \tag{19}$$

Δ_t^n is a random variable and to impose the debt stock constraint we need again a risk measure. We can bound the worst-case, or the risk-neutral, or the coherent risk measure of stock changes. We chose a coherent risk formulation consistent with the aggregate risk measure for flow, so that debt stock is non-increasing at the α confidence level.[10] Following again Krokhmal et al. (2002); Rockafellar and Uryasev (2000, 2002) we model (18) on the tree, using the linear system for all states $n \in \mathcal{N}$,

$$\begin{aligned}
\Delta_t^n &= d_t^n - d_{t-1}^{a(n)} && (20)\\
\Delta^{\diamond\diamond} &= \Delta^{\diamond} + \frac{1}{1-\alpha}\sum_{n\in\mathcal{N}} p^n y^n && (21)\\
y^n &\geq \Delta_t^n - \Delta^{\diamond} && (22)\\
y^n &\geq 0 && (23)\\
\Delta^{\diamond\diamond} &\leq \delta. && (24)
\end{aligned}$$

$\Delta^{\diamond}$ is the Value-at-Risk of debt stock changes, and $\Delta^{\diamond\diamond}$ is the conditional Value-at-Risk.With this formulation, the changes are bounded by δ at the α confidence level.

We note the risk interpretation of the debt stock constraint. By changing δ parametrically together with ω we can trace a three-dimensional efficient frontier trading-off refinancing and debt stock risks with cost. Our baseline model optimizes the trade-off between gross financing needs and costs, and treats debt stock with a (probabilistic) constraint.[11]

Smoothing of issued maturities

The baseline model specified so far does not impose any constraint on the issuance strategy in terms of its smoothness over time or its starting/ending points. Nonetheless, in some cases, introducing these constraints improves the stability of model results and the real-life practical implementability of the optimal strategies. On the demand side, there is a need for stability in the debt instruments issued by a sovereign so that there is always a complete term structure of sovereign debt. From the supply side, there are advantages from taxation and deficit smoothing. We set these constraints below.

If M_j denotes the maturity of the jth instrument, the *weighted average maturity of issued debt* at t under an adaptive fixed-mix strategy is given by

$$WAMI_t = \sum_{j=1}^{J} w_t(j) M_j. \tag{25}$$

[10] Worst-case and risk-neutral formulations are in Appendix A.

[11] We also considered some alternative specifications for the objective function and the constraints. For instance, we experimented with a model that optimizes the trade-off between expected debt stock level with tail risk of debt stock dynamics, and treats gross financing needs with a (probabilistic) constraint.

The restriction

$$| \; WAMI_t - WAMI_{t-1} \; |\leq \lambda, \tag{26}$$

where λ is a user specified parameter, limits changes of the average maturity relative to that issued at the previous period.[12]

Boundary conditions

We can also specify boundary conditions. For instance, we may want to impose that the sovereign starts with a WAMI close to the weighted average maturity of the legacy debt k_0, and finishes at the risk horizon with a specific target k_T,

$$WAMI_0 = k_0, \;\; WAMI_T = k_T. \tag{28}$$

Smoothing and boundary conditions can be applied to adaptive fixed-mix and dynamic strategies. Fixed-mix strategies are by definition smooth, and imposing a boundary condition trivially specifies the solution.

4.4 Closing the model

Debt dynamics accounting identities

To complete the model we give the accounting identities for debt dynamics, for all states $n \in \mathcal{N}_t$, at each time period $t = 0, 1, 2, \ldots T$. Again, some ingenuity is needed to exploit the tree structure, and we use an indicator function $\mathbb{1}^n(j, m)$ to keep track of maturing endogenously created debt,

$$\mathbb{1}^n(j,m) = \begin{cases} 1, \text{ if instrument } j \text{ issued at state } m \text{ matures at state } n, \\ 0, \text{ otherwise.} \end{cases} \tag{29}$$

[12]The absolute value function is not continuously differentiable but in the context of our model we can stay in the realm of continuous optimization by introducing variables v_t^+ and v_t^- to denote, respectively, increase and decrease of weighted average maturity at t

$$v_t^+ \geq WAMI_t - WAMI_{t-1}, \text{ and } v_t^- \geq WAMI_{t-1} - WAMI_t. \tag{27}$$

Constraining these variables to be non-negative, we get only one of the two to be non-zero, and its value will be the absolute value of average maturity changes, which we then bound by λ, i.e., $0 \leq v_t^+, v_t^- \leq \lambda$.

The flow dynamics (cf. eqn. 1) are written as

$$GFN_t^n = \underbrace{I_t^n + A_t^n}_{\substack{\text{Legacy service}\\ \text{payments}}} - \underbrace{PB_t^n}_{\substack{\text{Primary}\\ \text{balance}}} \tag{30a}$$

$$+ \underbrace{\sum_{m \in \mathcal{P}(n)} \sum_{j=1}^{J} X_{\tau(m)}^m(j)\, CF_t^n(j,m)}_{\substack{\text{Interest payment}\\ \text{of debt financing decisions}}} \tag{30b}$$

$$+ \underbrace{\sum_{m \in \mathcal{P}(n)} \sum_{j=1}^{J} X_{\tau(m)}^m(j)\, \mathbb{1}^n(j,m)}_{\substack{\text{Principal amortization}\\ \text{of debt financing decisions}}} . \tag{30c}$$

Comparing with (1) we have I_t^n as the part of $i_{t-1}D_{t-1}$ due to legacy debt and (30b) due to endogenously created debt, by our financing decisions. Similarly, A_t^n is the part of A_t due to legacy debt and (30c) due to financing decisions.

The *debt stock dynamics* are given by the recursive equation

$$D_t = (1 + i_{t-1})D_{t-1} - PB_t. \tag{31}$$

They can also be expressed in terms of flows, and on the scenario tree we have

$$D_t^n = D_{t-1}^{a(n)} + GFN_t^n - \sum_{m \in \mathcal{P}(n)} \sum_{j=1}^{J} X_{\tau(m)}^m(j)\, \mathbb{1}^n(j,m) - A_t^n. \tag{32}$$

Substituting (30) into (32) we link debt financing decisions to the effective interest rate on debt, which was the point of departure for our model,

$$D_t^n = D_{t-1}^{a(n)} + I_t^n - PB_t^n + \sum_{m \in \mathcal{P}(n)} \sum_{j=1}^{J} X_{\tau(m)}^m(j)\, CF_t^n(j,m). \tag{33}$$

Comparing this with (31) we get the *effective cost of debt* i_t at state n as

$$i_t^n = \frac{I_t^n + \sum_{m \in \mathcal{P}(n)} \sum_{j=1}^{J} X_{\tau(m)}^m(j)\, CF_t^n(j,m)}{D_t^n}. \tag{34}$$

The numerator is the net interest payment optimised in the objective function (5).

Baseline model specification

The complete baseline model consists of the objective function and decision variable definitions eqns. (5)–(11), flow risk constraints (13)–(16), stock risk constraints (20)–(24), smoothing constraints (25)–(26), boundary conditions (28), flow dynamics (30), and stock dynamics (33). We add non-negativity constraints $w_t(j) \geq 0$ for all t and j to avoid short sales.

This is the adaptive fixed-mix model formulation. To optimize dynamic strategies replace the time-dependent and state-invariant variables $w_t(j)$ by time- and state-dependent $w_t^n(j)$. For fixed-mix strategies replace the variables by time- and state-invariant $w(j)$.

The baseline model is flexible to allow for constraints that are justified either by economic theory or by practical considerations. It can be extended to model foreign currency debt (Bohn, 1990), upper and lower bounds on instruments of different maturities (Perold, 1984), inter-temporal smoothing of gross financing needs, or political or normative "principles-based constraints" (Guzman and Stiglitz, 2016).

5 Model calibration

We calibrate the model on a hypothetical but realistic economy, based on eurozone crisis countries. In the baseline specification, this economy exhibits a 3.5% long-term nominal growth and a 1% primary surplus. Our baseline assumptions imply sustainable debt dynamics in the long run, but we also tested exhaustively *non-sustainable cases* —by keeping all other country characteristics fixed and changing the long term expected growth to 3%— as we conducted systematic sensitivity checks and controlled experiments on all the relevant variables and parameters of the model. The model is calibrated for the forty-year period 2019–2059, to extend past 2049 when all legacy debt matures (Section 5.2).The available instruments are 3-year (short-term), 5-year (medium-term) and 10-year (long-term) bonds. We use optimizer BARON from (GAMS Development Corporation, 2016) to fit the trees and CONOPT (Drud, 1985) to solve the model.

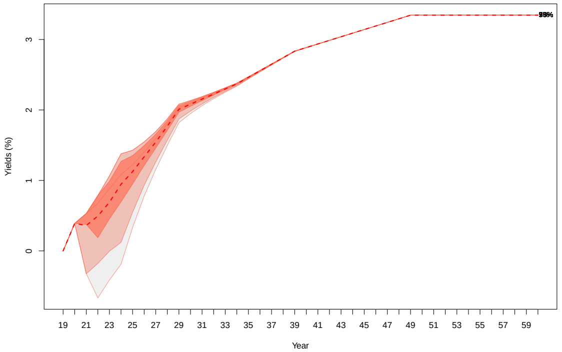
5.1 Scenario tree

There have been significant advances in the calibration of scenario trees to match market observed moments for multiple risk factors. Notably, Høyland and Wallace (2001); Klaassen (2002) developed calibration methods for use in stochastic programming models. Consiglio et al. (2016a) developed an arbitrage-free calibration procedure for both risk-neutral and objective probabilities to match an arbitrary number of moments, which is well suited for fitting trees to risk-free rates, GDP growth, and primary balance.

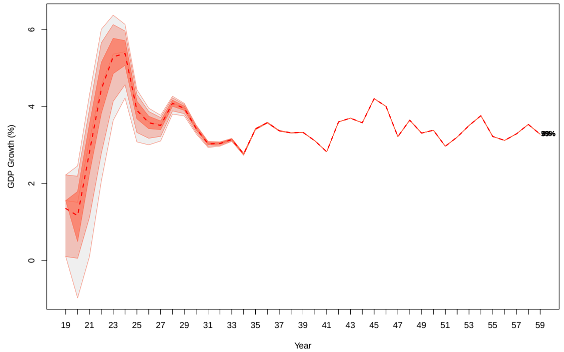
We calibrate the tree by adding to the long-term expected values, time-dependent random shocks. The tree is calibrated by matching the standard deviations and correlations of Figure 4(a), which mostly reflect European historical patterns. For tractability —the number of nodes in a tree grows exponentially with time— we calibrate a tree structure for five years for a total of 256 scenarios, with each scenario extended past the fifth year until the risk horizon using cyclical dynamics with normally distributed random shocks. Fig. 4(b)-(d) show the three calibrated trees of our baseline economy.

	Growth	Primary balance	Risk-free rates
Growth	1.00	0.25	-0.20
Primary balance	0.25	1.00	-0.03
Risk-free rates	-0.20	-0.03	1.00
Long-term mean	3.5	1.0	3.25
St. Dev	0.75	0.15	0.85

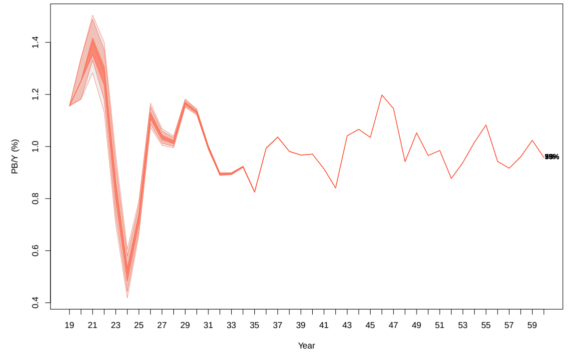
(a) Data to calibrate the scenario tree



(b) Scenarios of risk-free rates



(c) GDP growth scenarios



(d) Primary balance scenarios

Figure 4 – Data to calibrate the baseline economy and the scenarios.

5.2 Legacy debt

In the baseline calibration, we assume that legacy debt amortizes over the period 2019-2044 according to the pattern of Figure 5. In our sensitivity analysis, we also explore alternative profiles with shorter horizons.[13]

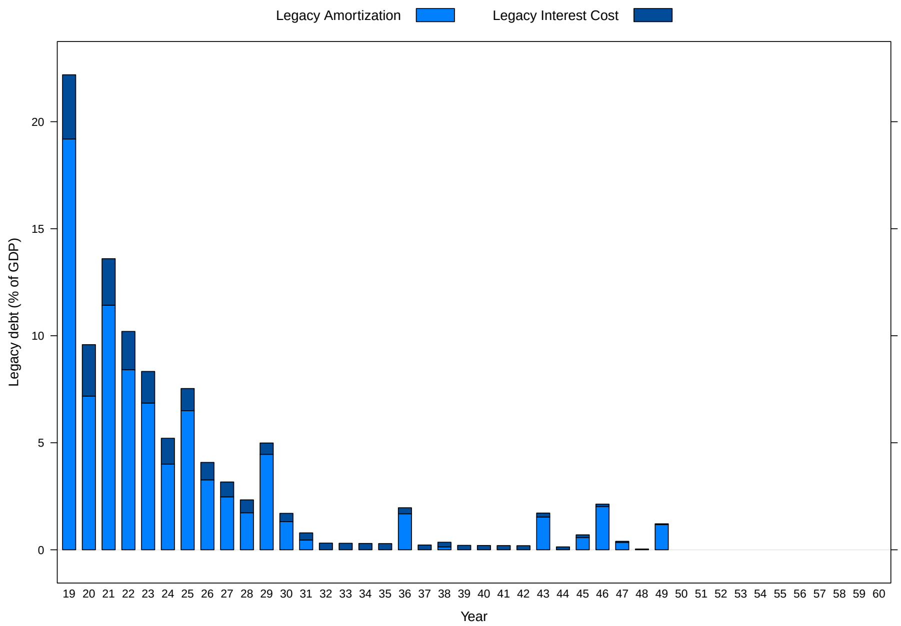


Figure 5 – Legacy debt service profile for the baseline economy.

5.3 Yield curve

Taking into account state dependence, eqn. (3) reads:

$$r_t^n(j) = r_{ft}^n + \rho(d_t^n, j). \tag{35}$$

Figure 4 already provides the risk-free rates on the scenario tree. Now we assume that the risk and term premia in our simulated economy behave such that

$$\rho(d_t^n, j) = a_j + (1 + b_j)\hat{\rho}(d_t^n). \tag{36}$$

where a_j and b_j are maturity-specific constants, and $\hat{\rho}(d)$ is a non-linear function that captures the endogeneity of interest rates to debt stock,

$$\hat{\rho}(d) \doteq \hat{\rho}\left[\frac{d_{max} - d}{1 + \exp(d_{max} - d)} - \frac{d_{min} - d}{1 + \exp(d_{min} - d)}\right]. \tag{37}$$

Together, eqns. (36) and (37) generate yield curves that shift and twist with changes of the debt ratio. For instance, if $\beta_j = 0$ for all maturities, changes in debt levels only cause parallel shifts in the yield curve but its shape remains unchanged and fully determined

[13]We also tested the model with redemptions evenly distributed over time or clustered around various specific time periods.

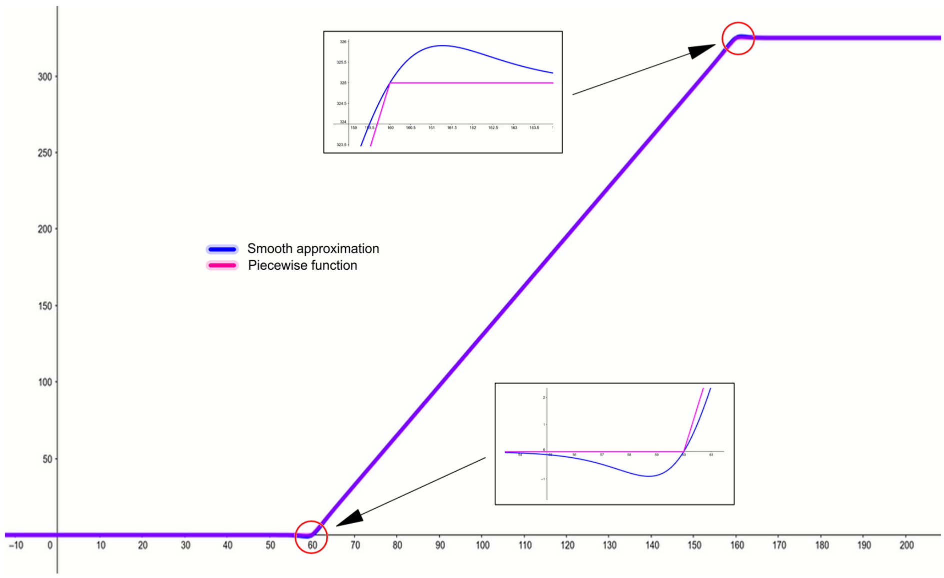


Figure 6 – Risk premium as a smooth approximation of a piece-wise function at $d_{min} = 60$ and $d_{max} = 160$.

by the a_j's. In contrast, when b_j is higher (lower) for long-term than short-term debt, an increase in debt also causes a steepening (flattening) of the curve.

The function in eqn.(37) is a smooth nonlinear approximation of the simpler, but non-continuously differentiable, piece-wise step function, Figure 6. The piece-wise function considers that a sovereign's risk premium is zero for debt ratios below 60%, and grows linearly with slope 3.25 for higher debt values, up to a peak of 325 basis points when the debt ratio is greater or equal than 160%. The implicit assumption in this function is that the sovereign is cut off the market for debt ratios above 160%, in which case its funding costs are stabilized by official sector support.

Table 1 reports the values of the parameters in eqns. (36)–(37) for the baseline calibration of the model. They are based on the yield curve dynamics observed in the European periphery during the last euro area debt crisis.

6 Model at work

We now test the model on our calibrated economy to answer three questions. First, what are the trade-offs embedded in choosing the optimal issuance strategy. Second, what is the economic relevance of these trade-offs. And third, why, how and when optimizing matters. We base our conclusions on the findings from adaptive fixed-mix strategies to answer the first two questions. Results for fixed-mix and dynamic strategies do not provide any different qualitative conclusions for these questions, but, quantitatively, there is a larger scope in trade-offs with dynamic strategies and more narrow scope with the fixed-mix rules. To illustrate these differences and answer the third question, we compare the most relevant variables in the model under the three alternative optimizing strategies and some simple issuance rules. Also, we do not always impose the stock risk constraints (20)–(24)

(a) Equation (36)

Coefficient	Bond maturity (years)		
	$j = 3$	$j = 5$	$j = 10$
a_j	-35bp	0	+72bp
b_j	-0.13	0	+0.13

(a) Equation (37)

Coefficient	Value
$\hat{\rho}$	3.25
d_{min}	60
d_{max}	160

Table 1: Parameters for the calibration of endogenous yield curves.

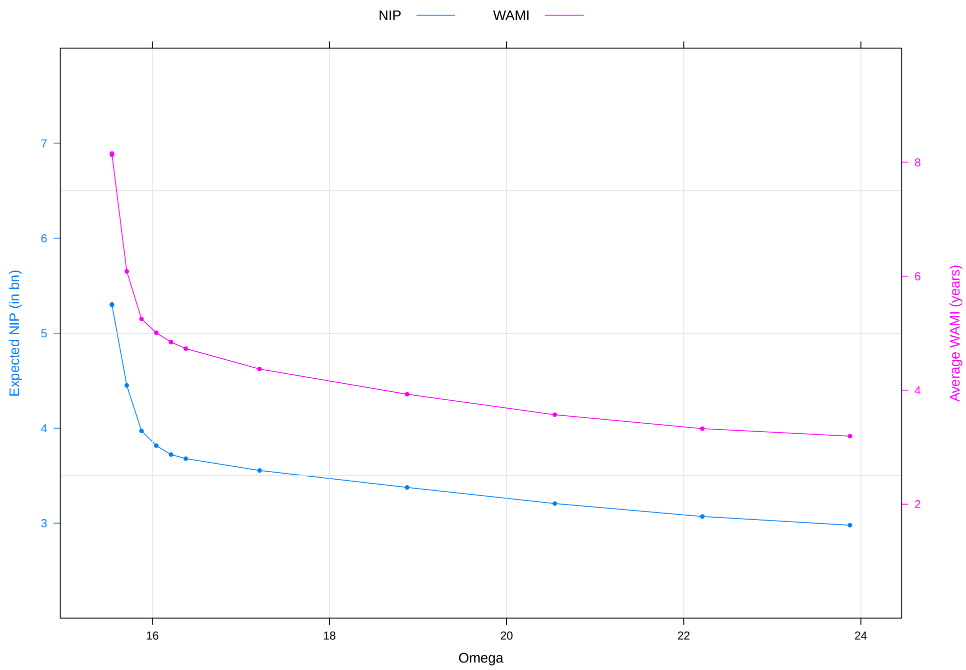
in order to better understand non-sustainable debt dynamics. In all the exercises, the confidence level for CVaR is set at $\alpha = 0.05$. In the results we also consider an upper threshold of ω in the range 15% to 20% GDP, which is the empirically observed market capacity to refinance a sovereign's debt. Larger values are of course possible, but the resulting financing strategies may not be implementable in practice.

We report various statistics of the solution as they vary with increasing risk tolerance ω, and also look at the trajectories of *gfn* and d, the WAMI of issued debt and its effective cost. For stochastic quantities we occassionally provide fan charts with the median and selected percentiles.

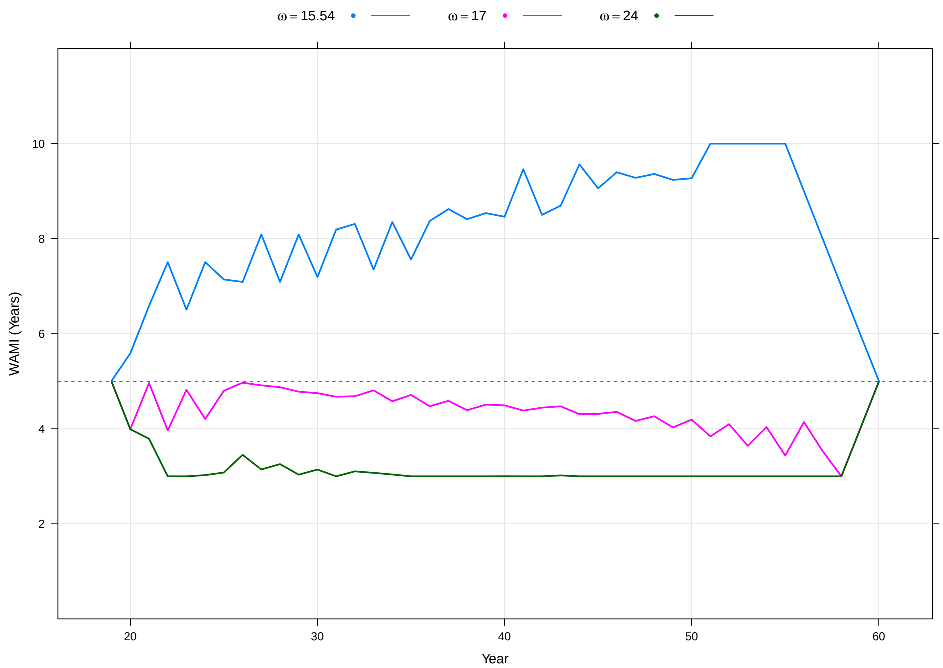
6.1 Trade-offs embedded in choosing optimal issuance strategy

The trade-off between cost of debt financing and refinancing risk has significant implications for the choice of an optimal issuance strategy, and the associated stock and flow dynamics. When going from the extreme case of cost minimization to minimizing refinancing risk we obtain more diversified financing strategies with less volatile gross financing needs that impose smoother demands on the tax base. However, stock declines more slowly. We quantify these trade-offs by varying ω from the lowest possible value for which the model has a feasible solution, to high values such that the risk constraint eqn. (6) is not binding. Results are summarized in Figures 7 and 8.

Remark 1. Risk management comes with a cost. Figure 7(a) shows the model's "efficient frontier" when issuance is designed to minimize expected interest payments on new debt subject to constraints on the tails of the stochastic distribution of *gfn*. From the figure, it is clear that reduced refinancing risk (i.e., lower ω) implies higher expected interest payments under an upward-sloping yield curve. This is so because more risk averse sovereigns will optimally choose issuance strategies that resort more often to long-term financing instruments which are more expensive. This is evident from the same figure, which shows the mean WAMI on the tree under the optimal issuance strategy for different values of ω. Figure 7(b) delivers the same message by comparing the evolution of WAMI over time for three different values of ω. The shift from long-term to short-term issuance as risk tolerance increases is consistent with the findings of Cole and Kehoe (2000); Conesa and Kehoe (2014). This shift creates even higher risks when a country is in trouble, leading Conesa and Kehoe (2015) to call it *gambling for redemption*, and our model results are consistent with their argument as risk tolerance increases.



(a) Expected net interest payments and weighted average maturity at issuance, averaged over the tree.

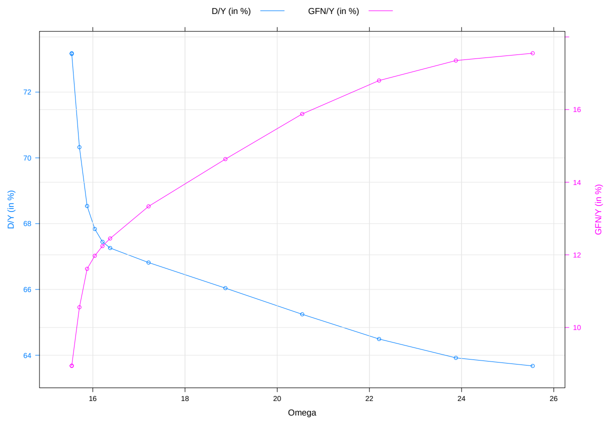


(b) Dynamics of the weighted average maturity at issuance.

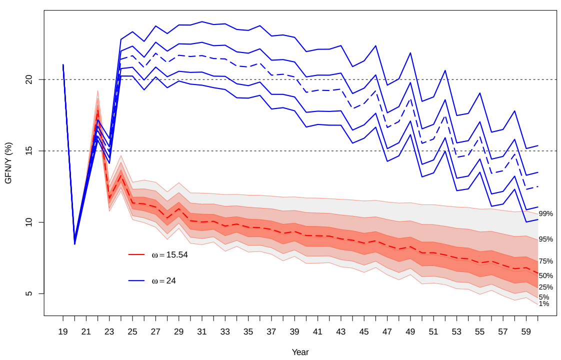
Figure 7 – The cost of risk management.

Remark 2. Trading off debt flow and stock dynamics. Besides the cost-risk trade-off described above, which could have been intuitively inferred from the mathematical formulation of our optimization problem, our simulations point at an additional trade-off. Namely, there is a conflict between improving the dynamics of gross financing needs and those of debt stocks. This is illustrated in Figure 8(a), which shows the average debt stock and the average gross financing needs over the tree under the optimal issuance strategy for different values of the risk tolerance parameter. Figures 8(b)–(c) convey the same message exploiting, for two different values of ω, all the temporal dimension and stochasticity of our model. The channel through which this trade-off operates relies, again, on the maturity and cost of the different financing instruments. As discussed above, it is possible to improve gross financing needs dynamics by issuing longer-dated bonds. But these are more expensive than short-term bonds, which increases the effective interest rate of debt and, subsequently, deteriorates debt dynamics.

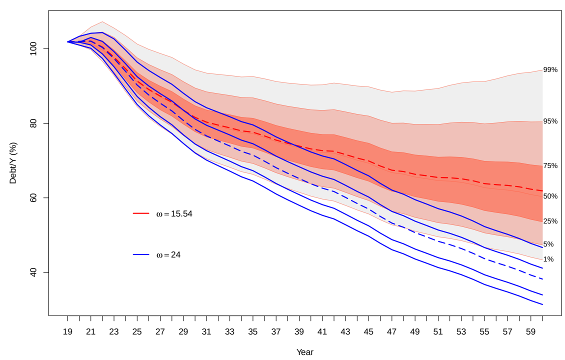
The fact that gross financing needs are smoother when risk tolerance is lower (Figure 8(b)) could intuitively imply smoother sovereign demands on the tax base. This result mirrors the conclusion of Missale (1997) that "the optimal taxation approach follows from a specification of [the] trade-off which gives all the weight to risk minimization".



(a) Debt and gross financing needs, averaged over the tree.



(b) Dynamics for gross financing needs (% of GDP).



(c) Debt stock dynamics (% of GDP).

Figure 8 – Trading off flow and stock dynamics for different levels of risk tolerance.

6.2 The economic relevance of trade-offs

Besides incorporating trade-offs that are relevant for choosing an issuance strategy, one important value added of our model is that it can quantify their magnitude. Reducing refinancing risks is always desirable, but when can this be too costly? How much should a Treasury increase the weighted average maturity of its issuances to reduce tail refinancing risks by 1%? Is the relationship between these variables linear? Addressing these matters without a rich and realistic quantitative tool may generate misleading policy advice. The model provides some insights into these issues.

Remark 3. The cost-risk and flow-stock trade-offs embedded in issuance decisions are key determinants of the evolution of debt dynamics and are economically significant. Conceptual trade-offs are only pertinent for policy makers as long as they have material quantitative effects. Table 2 shows that this is the case for our realistic calibrated economy. In particular, we find that reducing risk tolerance from a relatively high level ($\omega = 26\%$ of GDP) to the lowest attainable level ($\omega = 15.5\%$ of GDP) implies about 5 years increase in the weighted average maturity of issuances and an increase in debt's effective interest rates of 0.8 percentage points (pp) on average over the tree. Consistent with these effects, gross financing needs drop by about 8pp while debt deteriorates by 9pp.

ω	WAMI (in years)	Effective rate (in %)	GFN (in % of GDP)	Debt (in % of GDP)
min=15.5	8.2	3.4	9.2	73.9
16	5.0	2.9	12.2	68.6
17	4.4	2.8	13.5	67.7
19	3.9	2.8	14.8	66.9
22	3.3	2.7	16.9	65.4
24	3.2	2.6	17.4	64.8
26	3.1	2.6	17.6	64.6

Table 2: The effect of risk tolerance on weighted average maturity (averaged over the tree) at issuance, effective interest rate on debt, gross financing needs, and debt ratio.

Naturally, the magnitude of these impacts depends on the specific calibration of the economy. As a reference, Figure 9 shows that the sensitivity of WAMI to risk tolerance increases with the initial stock of debt. The same applies to net interest payments and the effective interest rate on debt (not shown).

Changing risk tolerance also has large distributional implications. Figures 10(a)-(b) show how the distribution of gross financing needs and debt stocks shift as risk tolerance drops from a relatively high level to the lowest attainable level. Figure 10(c) complements these results by showing the change of the standard deviation of these variables over the tree. Again, risk management has a significant quantitative impact.

Remark 4. Risk management comes with some limits and non-linearities. In the paragraphs above we talked about the "lowest" risk tolerance level. This is the lowest value of ω for which the model has a feasible solution. This threshold, which depends on the precise calibration of the economy, is already informative by itself as it

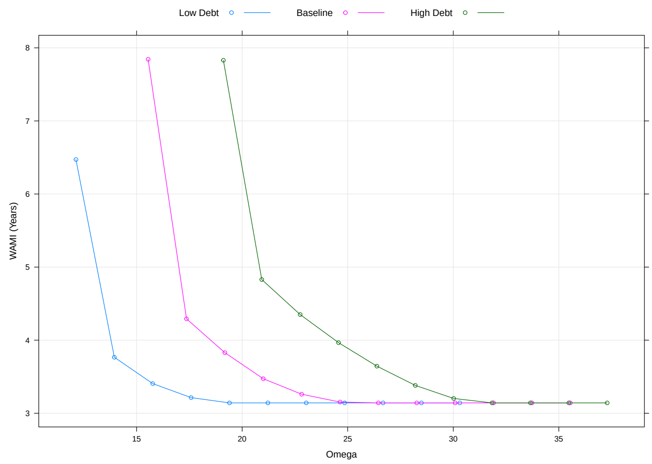
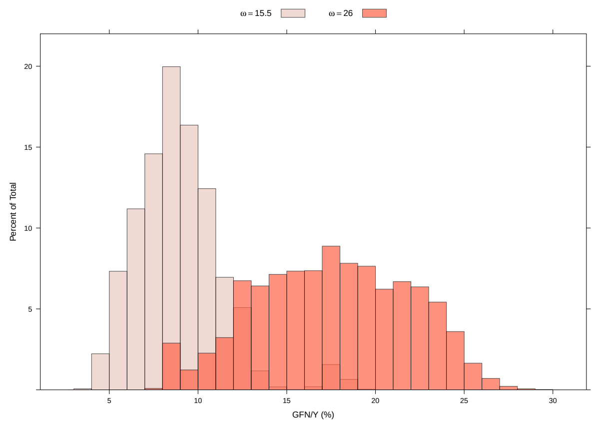
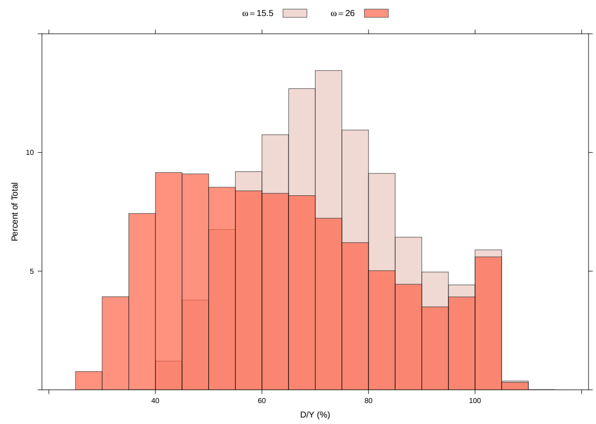


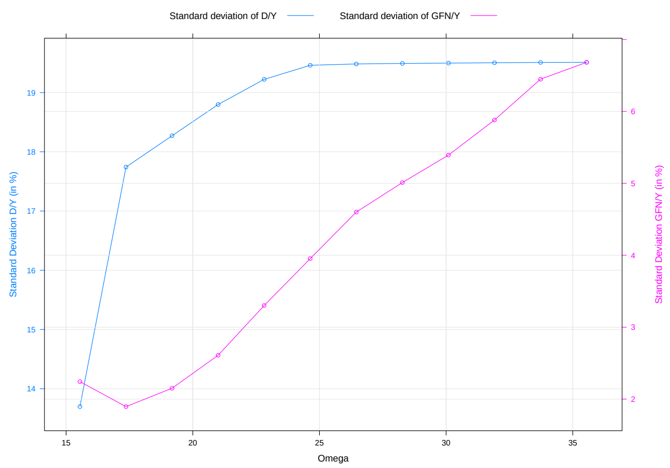
Figure 9 – Weighted average maturity, averaged over the tree, at issuance, for initial debt stock levels deviating by ± 20% from the baseline economy.



(a) Debt flow distribution over the tree (GFN as % of GDP)



(b) Debt stock distribution over the tree (% of GDP)



(c) Standard deviation of debt flow and stock over the tree (% of GDP)

Figure 10 – Distributional implications of risk tolerance on debt dynamics.

provides a clear benchmark for assessing debt sustainability. In particular, knowing this minimum (unavoidable) level of exposure to tail refinancing risks can preclude the quest for untenable policy targets.

Our quantitative analysis also unveils some marked non-linearities in the risk management of issuance strategies. Intuitively, when risk tolerance is very high (i.e., high ω), the CVaR constraint on gross financing needs is barely binding (if at all). In those cases, a given reduction in ω will have relatively small effects on issuance, cost, flow and stock dynamics, which will be mostly driven by cost minimization. In contrast, when the initial value of ω is already small, the same reduction in tail risks will imply larger distortions in the relevant variables of the problem. These non-linear effects are particularly evident for WAMI and debt stocks (Table 2). In our simulation, reducing risk tolerance from 26% to 19% of GDP increases effective rates marginally (0.2pp) and requires a maturity extension of 0.8 years. However, reducing risk tolerance from 17% to the lowest possible15.5% has a marginal impact on costs and maturities three and five times larger, respectively.

6.3 The relevance of optimizing

In this section we compare the performance of different optimizing strategies (fixed-mixed, adaptive fixed-mixed, and dynamic) and various rules-of-thumb. Of course, we know from optimization theory that results will always be better, for our optimality criterion, with the more flexible dynamic issuance strategies. The economic significance of this outcome depends on the calibrated economy, so in this section we seek to understand how optimizing matters and when it becomes more important.

Remark 5. Cost savings from (flexible) optimization increase as risk tolerance declines. Figure 11 shows the cost-risk trade-off in our simulated economy for our three optimizing strategies and for three simple issuance rules: issuing always long-term (i.e., using 0-0-100 weights for the 3y-5y-10y funding instruments), issuing always short-term (i.e., 100-0-0) and a benchmark issuing always in all tenors with a weighted average maturity of about 5 years (i.e., 40-40-20). As expected, for each level of risk tolerance, dynamic strategies are always the "cheapest", and non-extreme simple rules, such as the benchmark 40-40-20, underperform the optimal ones (except by serendipitous coincidence). The relative performance of extreme simple rules is also intuitive. When risk tolerance is very high and the sovereign only cares about cost minimization, it always issues at the shortest available tenor and the optimal issuance strategy coincides with the 100-0-0 rule. We also found that when the service profile of legacy debt is smoothly decreasing and risk tolerance is very low, issuing always at the longest available maturity (i.e., the 0-0-100 rule) is the most efficient fixed-mix strategy to limit refinancing risks. Except in these special marginal cases, Figure 11 shows that cost savings from optimization are economically meaningful and that the relative benefits of the more flexible optimization approaches increase non-linearly as risk tolerance declines.

Remark 6. Optimizing renders less volatile financing needs but weighs on debt stock dynamics. Remarks 1 and 2 above already discussed the cost-risk and flow-stock trade-offs arising from an optimal issuance model with adaptive fixed-mixed strategies. We now illustrate how other optimal strategies and simple rules handle the same trade-offs. For expositional purposes, we compare the behaviour of the 40-40-20 benchmark rule which serendipitously is the optimal fixed-mix strategy for the calibrated economy, with unconstrained adaptive fixed-mix strategies without boundary conditions.

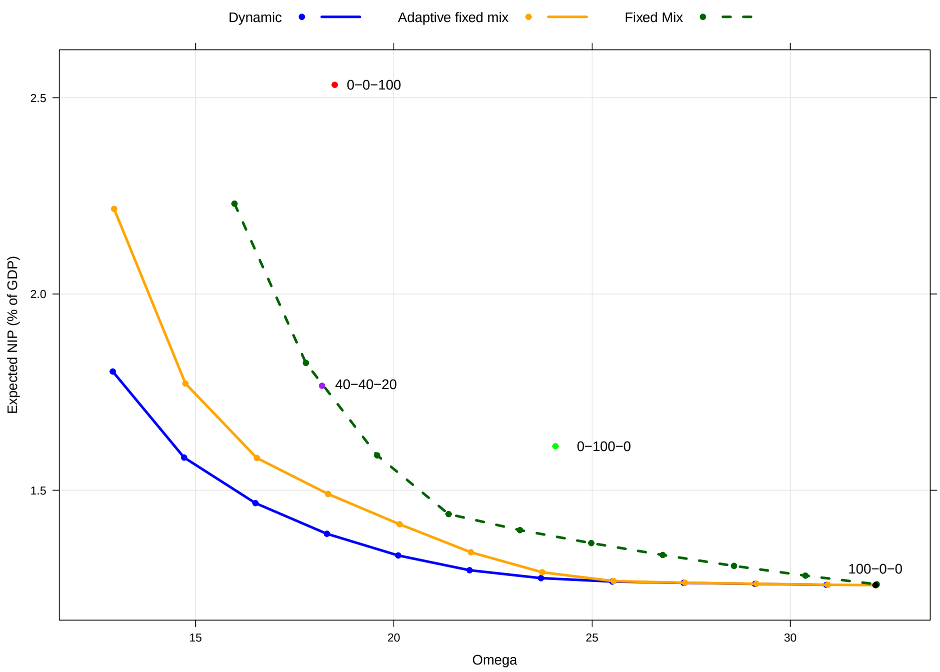


Figure 11 – Expected net interest payment and risk tolerance for different issuance strategies.

Dynamic strategies dominate adaptive fixed-mix which dominates fixed-mix and several simple rules. Differences are bigger with decreasing risk tolerance and the trade-offs are economically significant. Serendipitously, the benchmark policy lies on the efficient frontier of the fixed-mix strategies at an intermediate risk preference.

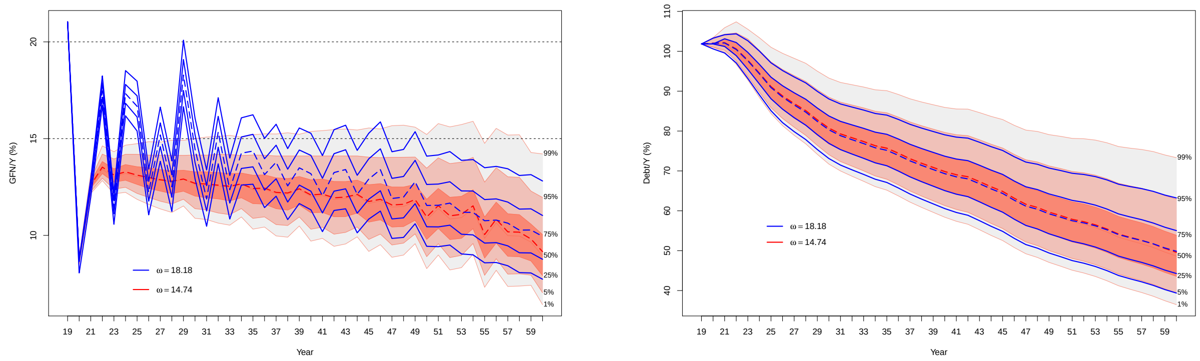
From Figure 11 we note that there are many ways to make a Pareto move from the point 40-40-20 on the frontier of the fixed-mix strategies, to the frontier of the adaptive fixed-mix strategy. We consider a move to the left, reducing ω from 18.18% to 14.74% of GDP. Figure 12(a) shows that the more flexible strategy has a lower volatility in gross financing needs, but debt stock declines at a somewhat slower pace, Figure 12(b).[14]

Large spikes in gross financing needs may derail debt dynamics, for instance, because temporary liquidity problems may weigh on long-term solvency. Optimal issuance strategies are able to smooth out some of these spikes and this makes them especially suitable for sovereigns with debt sustainability concerns. For instance, from Figure 12(a) we observe that the optimal strategy keeps financing needs below the 15% threshold after 2022 with high probability, and has smooth payments. The average needs from the benchmark rule violate the threshold six times during the first decade, and there is a 0.05 probability of more violations until 2049 when all legacy debt expires.

Of course, debt-financing decisions may not restore (by themselves) the sustainability of explosive debt dynamics. But in marginal cases, optimizing certainly renders significant improvements, and we address the issue of restoring sustainability with adjustments of gross financing needs in Section 7.

Remark 7. Optimizing helps relatively more when the stock of legacy debt is larger and its maturity shorter. Ceteris paribus, more legacy debt means worse

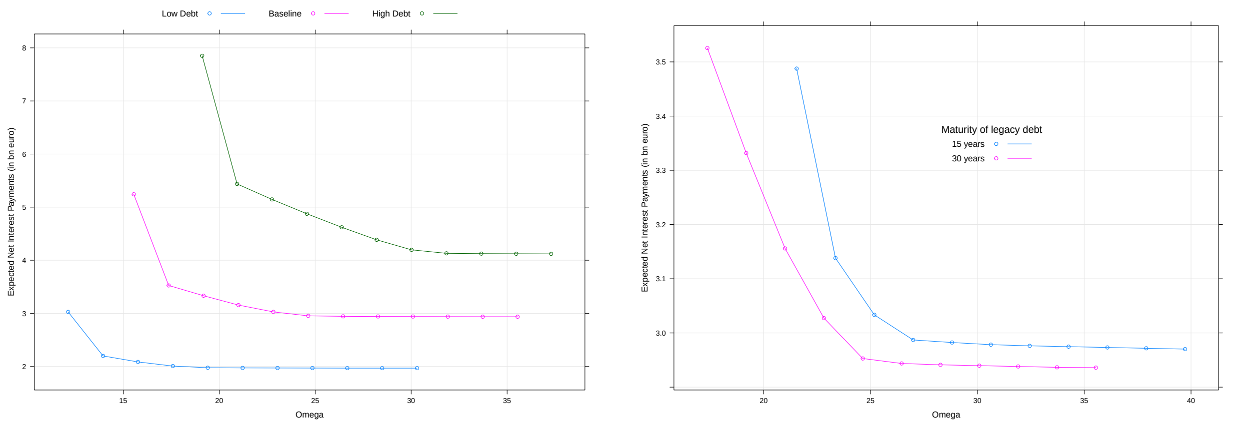
[14] Alternatively, we could move down, and this would reduce expected NIP by about 0.5% of GDP and the effective cost of debt by 40bp, but will not improve refinancing risks.



(a) *gfn* **(b)** *d*

Figure 12 – Debt dynamics with the optimal adaptive fixed-mix strategy and the benchmark rule.

initial conditions, and this implies both higher average gross financing needs over the relevant horizon (i.e., higher refinancing risks), and higher funding costs due to interest rate endogeneity. Consistent with this intuition, Figure 13(a) shows that the efficient frontier for adaptive fixed-mix strategies shifts to the right when the stock of legacy debt increases. The same is true when the maturity of this debt shortens (Figure 13(b)) as, ceteris paribus, this means higher refinancing pressures for the sovereign as legacy debt amortizes on a shorter horizon.



(a) Expected net interest payment averaged over the tree, and ω for different initial stocks of debt.

(b) Expected net interest payment averaged over the tree, and ω for different maturities of legacy debt.

Figure 13 – Sensitivity of the relative benefits of optimization to initial debt stock and maturity.

More importantly, the benefits from using more flexible optimal strategies are relatively larger in the presence of worse initial conditions. Table 3 reports the lowest expected interest costs that can be achieved with fixed-mix and adaptive fixed-mix strategies for an arbitrary risk tolerance ($\omega = 20$) and different initial conditions of legacy debt and its maturity. The outperformance (in terms of lower interest costs) of the more flexible strategies is more evident in the worst scenarios. The same is true when comparing adaptive fixed-mix strategies and the 40-40-20 rule for the same level of risk tolerance, ω^*.

	Baseline	High debt	Short maturity
$\omega = 20$			
Adaptive fixed-mix (AFM)	3.22	5.79	3.86
Fixed-mix (FM)	3.57	8.17	5.21
Difference AFM-FM	-0.34	-2.38	-1.51
ω^*	18.19	22.82	22.74
Adaptive fixed-mix	3.41	5.09	3.16
Benchmark 40-40-20	4.12	6.15	4.20
Difference AFM-benchmark	-0.71	-1.06	-1.05

Table 3: Expected net interest payments, averaged over the tree, for different issuance strategies and calibrated economies.
ω^* is the risk tolerance of the optimal adaptive fixed-mix strategies when set equal to the level of risk of the benchmark rule in each of the calibrated economies, to make for a fair comparison.

Remark 8. The feedback from debt stock into interest rates affects risk management. As discussed above, our model's cost-risk trade-off works through the maturity of the funding instruments issued to meet ongoing gross financing needs. Ceteris paribus, higher risk aversion (to refinancing risks) means more reliance on long-term instruments, lower gross financing needs, but higher debt (Remarks 1 and 2).

Against this background, an increase in $\hat{\rho}$, the parameter that drives the sensitivity of funding costs to the stock of debt in eqn. (37), reduces the incentives to lengthen the maturity of the issuances. This implication is irrelevant of whether debt dynamics are fundamentally increasing or decreasing. Endogenous rates open the room for both vicious and virtuous debt-cost cycles. They always create an incentive to shorten the maturity of the issuances, either to accelerate the virtuous cycle or to mitigate the negative implications of vicious cycles.

As in Remark 7, higher sensitivity of interest rates to the debt level (i.e., stronger interest rate endogeneity) can be viewed as worse initial conditions for a treasury interested in reducing refinancing risks. This can be seen in Figure 14, which compares the efficient frontiers for two levels of $\hat{\rho}$. Two observations arise from these results. First, debt sustainability analysis that ignores interest rate endogeneity may lead to erroneous conclusions both quantitatively and qualitatively. This observation is consistent with the conjecture of Bohn (1990) that endogenous interest rates change not only quantitatively, but also qualitatively, the sovereign's optimization problem. Second, abstracting from moral hazard considerations, our model suggests that mitigating interest rate endogeneity would be positive both from a cost and risk management perspective. To a large extent, this is the aim of financial assistance programmes to countries facing debt sustainability problems.[15]

[15]The relative benefits of optimizing documented in Table 3 were also found to be bigger for stronger feedback loops, especially for higher debt and shorter maturities.

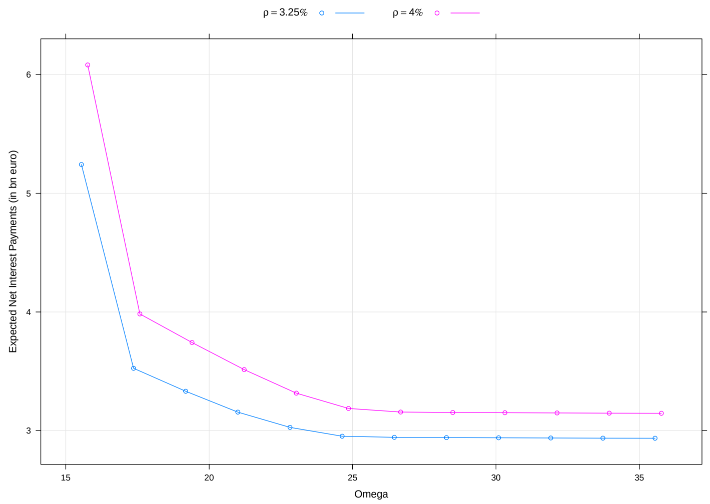


Figure 14 – Expected net interest payments, averaged over the tree, and risk tolerance under different sensitivities of interest rates on debt stock.

7 Adjusting gross financing needs for sustainability

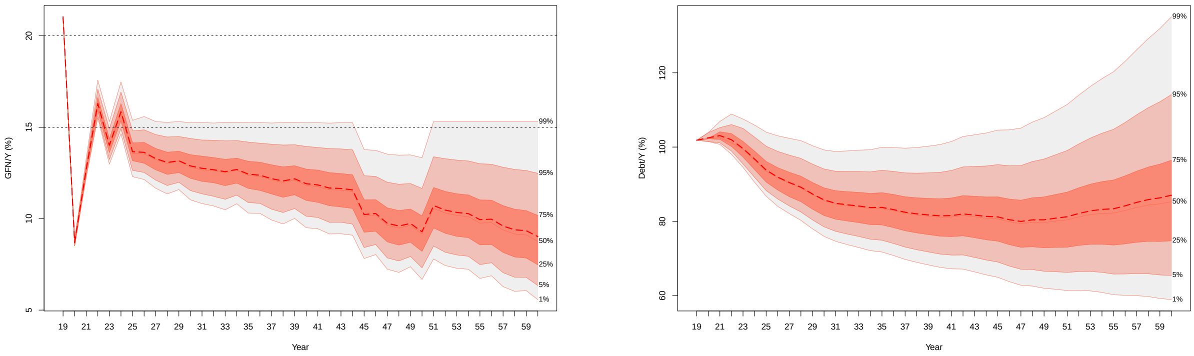
The risk constraints on gross financing needs (eqns. 13–16), and the pace of debt reduction (eqns. 20–24), are at the heart of our model. In these constraints, ω defines the policy marker's risk tolerance, and δ the desired pace of debt stock reduction. We have already hinted that, depending on the economic fundamentals of the economy, there is always a threshold for ω beyond which refinancing risks can not be reduced further, even by implementing the smartest issuance strategies (Remark 4). The same applies to δ.[16] An ambitious policy maker may want to see debt declining at a very fast pace, but that could be beyond the potential of the economy. As discussed above, quantifying these thresholds is already an important contribution of our model, which is especially convenient for debt sustainability discussions involving the official sector and sovereigns in debt crises. We now extend our baseline model to contribute further to this debate. In particular, the model extension provides both qualitative and quantitative responses to the following question: What can you do to ensure that the desired, but initially unattainable, targets of refinancing risks and pace of debt reduction are reached?

We identify the *hot spots* where adjustments may be required. Adjustments can mobilize a combination of domestic resources, such as higher revenues (e.g., tax proceeds, privatisations) or expenditure reductions, and external resources, such as official sector financing or debt restructuring.

To compute the adjustments required to reach a feasible solution in the constrained optimization problem, we introduce variable u_t to denote adjustments as a proportion of GDP. If the sovereign manages to save (or to raise) an additional amount $u_t Y_t^n$ at state n of period t, then the debt financing eqn. (9) reads as

$$\sum_{j=1}^{J} X_t^n(j) + u_t Y_t^n \geq GFN_t^n. \quad (38)$$

[16] Of course, the model is always feasible if we relax the stock constraint to allow debt to grow, and ignore refinancing risks.



(a) *gfn* **(b)** *d*

Figure 15 – Debt dynamics for an economy calibrated under non-sustainable conditions.

In this equation, $u_t Y_t^n$ represents the part of gross financing needs that does not need to be financed by issuing new debt. Of course, if u_t is unbounded and carries no cost, the model will always meet all financing needs through u_t. To ensure that these adjustments are used as a last resort, we add a penalty term $\mathcal{M} \sum_{t=0}^{T} u_t$ to the objective function (5), where the large constant $\mathcal{M}$ ensures that we compute the *minimum* amount required to meet the desired policy targets on refinancing risks and stock dynamics. The timing of these adjustments identifies the *hot spots*. A smoothing constraint can be imposed, similar to the smoothing of issued maturities.

Equation (38) holds with inequality because the adjustment u_t is time-dependent but state-invariant, whereas the total amount raised from a given adjustment is state-dependent due to GDP. This means that, under some states of the economy, a surplus could be created. We assume that this state-dependent surplus will be used to pay down debt, and the debt dynamics eqn. (32) is modified accordingly,

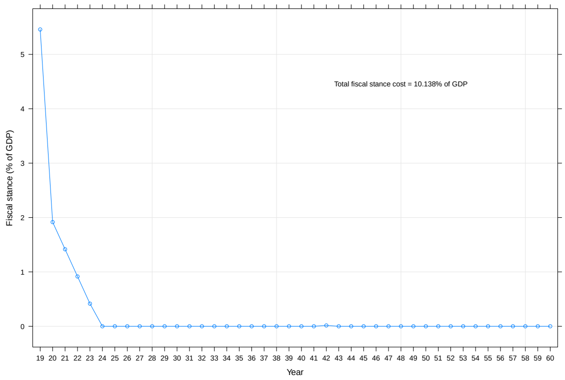
$$D_t^n = D_{t-1}^{a(n)} + GFN_t^n - u_t Y_t^n - \sum_{m \in \mathcal{P}(n)} \sum_{j=1}^{J} X_{\tau(m)}^m(j) \mathbb{1}^n(j,m) - A_t^n. \tag{39}$$

An alternative specification allows the adjustments to be both state- and time-dependent, u_t^n. In this case (38) holds with equality, and the optimal solution could be used to structure *contingent contracts* (Bazerman and Gillespie, 1999) for a country under an assistance program.

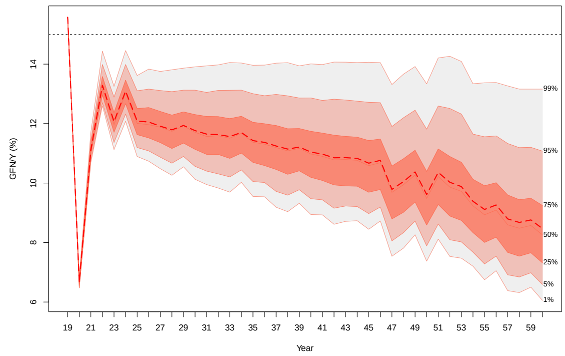
We revisit the calibrated economy, but we lower long-term expected GDP growth to 3% to create non-sustainable debt dynamics. The optimal adaptive fixed-mix strategies are now unable to reduce refinancing risks below 16% of GDP. Figure 15 shows the dynamics of gross financing needs and debt stock for this risk level, and comparing with the optimized dynamics of the baseline economy —Figure 8 for $\omega = 15.54$— we observe a significant increase of gross financing needs and explosive growth of debt stock.

We now search for the minimum adjustments that are required to reduce financing risks below the threshold of $\omega = 15\%$ of GDP. The minimum achievable risk level is $\omega = 13.7\%$ and Figure 16(a) shows the size and timing of the adjustments, and Figure 16(b) shows the new risk-complying gross financing needs that violate the threshold only at the first period, due to the large legacy debt maturing in 2019. The model suggests adjustments of more than 5% of GDP in the early periods for a total adjustment of 10.14% of GDP. If these were, for example, additional fiscal efforts, they may reflect a "surplus of ambition" (Eichengreen and Panizza, 2016), and may be difficult to implement in practice due to technical and/or political reasons. In the same vein, such adjustments through privatization proceeds may create concerns for fire sales under adverse conditions. To address such concerns, we re-run the model setting a cap of 3% of GDP per period in the adjustments, Figure 16(c)-(d). Naturally, this is a second best and, although the adjustment per year is smaller, it carries on for two more years, the total adjustment is slightly higher (10.52 % of GDP), and gross financing needs dynamics worsen (even if the aggregate ω stays the same). Debt stock dynamics (not shown) are decreasing with both adjustments.

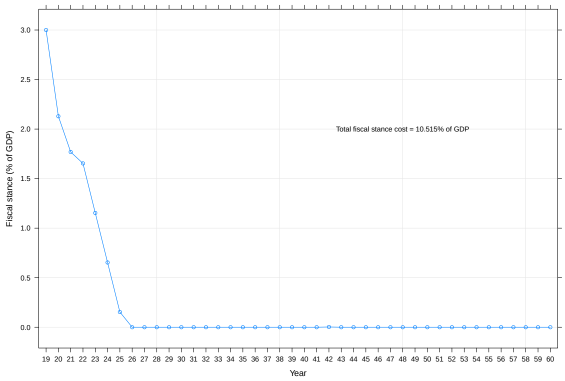
We also consider delaying the adjustments. In particular, we re-run the model under the assumption that no adjustments can be implemented in the first year of the decision horizon, i.e. $u_{2019} = 0$, see Figure 16(e)-(f). An important observation is that the total adjustment following the initial delay (12.55% of GDP) lasts for another two years and is significantly higher than the total required if the country does not procrastinate (10.52% of GDP), and gross financing needs dynamics worsen even further (although the aggregate ω stays the same). This result mirrors the finding from the stylized model of Blanchard et al. (1990) that "delaying adjustment substantially affects the size of the needed policy action". These findings can inform strategic policy decisions for public finance and operational decisions for official sector borrowing.



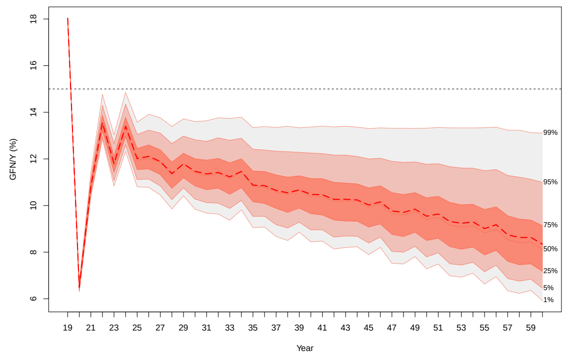
(a) Unrestricted adjustments



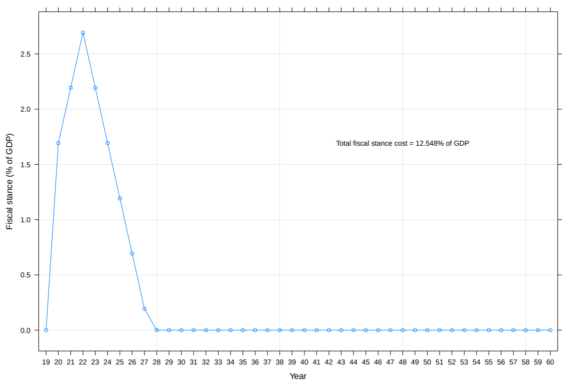
(b) *gfn* with unrestricted adjustments



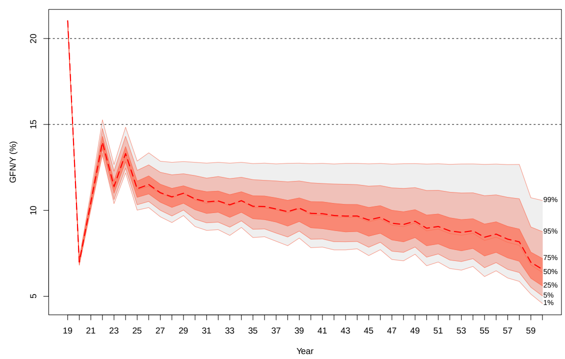
(c) Restricted adjustments



(d) *gfn* with restricted adjustments



(e) Delayed adjustments



(f) *gfn* with delayed adjustments

Figure 16 – *Hot spots* and adjustments of gross financing needs required to reach an acceptable (and sustainable) refinancing risk for an economy calibrated under non-sustainable conditions.

8 Conclusions and further work

In this paper, we have presented a very granular and flexible tool that, by incorporating the main trade-offs embedded in the design of optimal issuance strategies, provides interesting qualitative and quantitative insights for the analysis of sovereign debt dynamics. Large-scale stochastic programming on scenario trees is a versatile and effective tool for debt sustainability analysis.

We see at least three avenues for further work:

1. *Real life implementation.* In this paper, we have used a simulated but realistic economy to illustrate the qualitative and quantitative functioning of our model. An obvious next step would be to apply our optimizing tool to real-country data complementing, from a risk management perspective, the debt sustainability analysis already carried out with more standard models.

2. *Debt overhang.* The ability of a country to create a primary surplus also depends on debt overhang (Kobayashi, 2015; Reinhart et al., 2012), through the mechanism of "fiscal fatigue" (Barr et al., 2014). To address this issue we need to link primary balance scenarios with debt-to-GDP ratio within the model. Addressing this issue is not simply a question of developing the appropriate simulations through fiscal multipliers (see, e.g., (Barr et al., 2014, eqn. 5)). Instead, we would need to link primary balance scenarios with the endogenous debt-to-GDP dynamics within the model, fully internalizing the feedback loop $X \to D \to r \to Y \to PB \to X$. This extension would also make growth endogenous to debt.

3. *Sovereign contingent debt financing.* There is an ongoing debate on the merits of contingent debt for sovereigns, with a special focus on sovereign-CoCos and GDP-linked bonds (IMF, 2017). Incorporating such instruments into our model requires two extensions. First, to develop the appropriate simulation models linking debt payments to the appropriate risk factor, in order to calibrate the scenario trees. For instance, we need to link the service payments of GDP-linked bonds to GDP scenarios (Benford et al., 2016; Blanchard et al., 2016; Consiglio and Zenios, 2018), or those of sovereign-CoCos to the factor (potentially) triggering a standstill (Consiglio et al., 2016b). Second, to link the new scenario trees to debt financing decisions. To do so, our model must incorporate additional factors relating to contingent contracts, and the endogenous modelling of cashflow payments of discrete contingent debt in the case of sovereign-CoCos. This may require integer programming, thus leading to complex mixed-integer non-linear programs.

The extension of the model in Section 7 also raises interesting questions. If the required adjustments imply additional fiscal effort, we would need to model the feedback from fiscal effort to growth. If the adjustments imply debt restructuring, we must consider its impact on the sovereign's yield curve taking into account lenders' considerations through "principles-based constraints" (Guzman and Stiglitz, 2016), and strike a balance between the use of domestic and external resources. Our setup provides a fertile ground for further work on these relevant research and policy questions.

A Modeling worst-case and risk-neutral measures

The worst-case gross financing needs for all states at all time periods is defined by $\Psi(gfn) = \max_{n\in\mathcal{N}_t,\, t=0,1,2,\ldots T}\{gfn_t^n\}$, and this is a suitable measure for stress testing. Under risk neutrality we consider the expected value of gross financing needs at each period, and the risk measure is $\Psi(gfn_t) = \sum_{n\in\mathcal{N}_t} \pi_t^n gfn_t^n$ for all t. Constraint eqn. (16) on the coherent risk measure can be replaced by constraints on these risk measures. For the worst-case we have

$$gfn_t^n \leq \omega, \text{ for all } n \in \mathcal{N}_t,\ t = 0, 1, 2, \ldots T. \tag{40}$$

For the risk-neutral measure we have

$$\sum_{n\in\mathcal{N}_t} \pi_t^n gfn_t^n \leq \omega,\ t = 0, 1, 2, \ldots T. \tag{41}$$

Similarly we can formulate debt stock constraints (section 4.3) using worst-case and risk-neutral measures. The worst-case formulation adds the constraints

$$\Delta^n = d^n - d^{a(n)} \tag{42}$$

$$\Delta^n \leq \delta, \text{for all } n \in \mathcal{N}. \tag{43}$$

This requires that debt is reduced with pace δ from its ancestor state, for all time periods and states. The risk-neutral formulation requires that average debt is inter-temporally non-increasing and adds the following constraints

$$\Delta^n = d^n - d^{a(n)} \tag{44}$$

$$\frac{1}{N_t}\sum_{n\in\mathcal{N}_t} \Delta^n \leq \delta,\ t = 0, 1, 2, \ldots T. \tag{45}$$

With this formulation, situations where debt stock cannot decrease by δ temporarily, even by small amounts, are considered unsustainable. We consider, instead, that the average debt changes over the risk horizon should decrease with the target pace using the less restrictive condition

$$\Delta^n = d^n - d^{a(n)} \tag{46}$$

$$\frac{1}{N}\sum_{n\in\mathcal{N}} \Delta^n \leq \delta. \tag{47}$$

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